BOUNDED ULTRAIMAGINARY INDEPENDENCE AND ITS TOTAL MORLEY SEQUENCES

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Abstract. We investigate the following model-theoretic independence relation: $b \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}_A c$ if and only if $\mathrm{bdd}^{\mathrm{u}}(Ab) \cap \mathrm{bdd}^{\mathrm{u}}(Ac) = \mathrm{bdd}^{\mathrm{u}}(A)$, where $\mathrm{bdd}^{\mathrm{u}}(X)$ is the class of all ultraimaginaries bounded over X. In particular, we sharpen a result of Wagner to show that $b \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}_A c$ if and only if $\langle \mathrm{Autf}(\mathbb{M}/Ab) \cup \mathrm{Autf}(\mathbb{M}/Ac) \rangle = \mathrm{Autf}(\mathbb{M}/A)$, and we establish full existence over hyperimaginary parameters (i.e., for any set of hyperimaginaries A and ultraimaginaries b and c, there is a $b' \equiv_A b$ such that $b' \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}_A c$). Extension then follows as an immediate corollary.

We also study *total* $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}$*-Morley sequences* (i.e., A-indiscernible sequences I satisfying $J \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}_A K$ for any J and K with $J + K \equiv^{\mathrm{EM}}_A I$), and we prove that an A-indiscernible sequence I is a total $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}$-Morley sequence over A if and only if whenever I and I' have the same Lascar strong type over A, I and I' are related by the transitive, symmetric closure of the relation '$J + K$ is A-indiscernible.' This is also equivalent to I being 'based on' A in a sense defined by Shelah in his early study of simple unstable theories [9].

Finally, we show that for any A and b in any theory T, if there is an Erdős cardinal $\kappa(\alpha)$ with $|Ab| + |T| < \kappa(\alpha)$, then there is a total $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}$-Morley sequence $(b_i)_{i<\omega}$ over A with $b_0 = b$.

Introduction

A central theme in neostability theory is the importance of various kinds of 'generic' indiscernible sequences—usually with Michael Morley's name attached to them—such as Morley sequences in stable and simple theories, strict Morley sequences in NIP and NTP_2 theories, tree Morley sequences in NSOP_1 theories, and $\mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}$-Morley sequences in rosy theories. A very broad question one might ask is this: How generically can we build indiscernible sequences in *arbitrary* theories?

Over a model M, we can always extend a given type $p(x) \in S_x(M)$ to a global M-invariant type $q(x) \supset p(x)$ and then use this to generate a sequence $(b_i)_{i<\omega}$ satisfying $b_i \models q \upharpoonright M b_{<i}$ for each $i < \omega$. In some cases the particular choice of $q(x)$ matters, but typically these sequences are robustly generic. Sequences produced in this way have a certain property, which is that they are *based on M in the sense of Simon*; i.e., for any I and J with $I \equiv_M J \equiv_M b_{<\omega}$, there is a K such that $I + K$ and $J + K$ are both M-indiscernible. In NIP theories, the sequences with this property are precisely the sequences generated by an invariant type [11, Prop. 2.38]. Over an arbitrary set of parameters A, however, there may fail to be any indiscernible sequences based on A. In the dense circular order, for instance, there are no indiscernible sequences based on $\varnothing$. Other technical issues also arise

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when working over arbitrary sets, such as the necessity of considering Lascar strong types over and above ordinary types.

A notion of independence $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$ is said to satisfy *full existence* if for any A, b, and c, there is a $b' \equiv_A b$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{A} c$. Together with a common model-theoretic application of the Erdős-Rado theorem (Fact 1.2), this implies that for any A and b, one can build an $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$-*Morley sequence*, an A-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ satisfying $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{A} b_{<i}$ for each $i < \omega$ (assuming $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$ also satisfies right monotonicity). Model-theoretically tame theories often have full existence for powerful independence notions, such as non-forking, but this does fail in some notable tame contexts.

One independence notion that is known to satisfy full existence in arbitrary theories is that of *algebraic independence* [2, Prop. 1.5]: $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{a}}_{A} c$ if $\mathrm{acl}^{\mathrm{eq}}(Ab) \cap \mathrm{acl}^{\mathrm{eq}}(Ac) = \mathrm{acl}^{\mathrm{eq}}(A)$. A natural modification of this concept is *bounded hyperimaginary independence*: $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{b}}_{A} c$ if $\mathrm{bdd}^{\mathrm{heq}}(Ab) \cap \mathrm{bdd}^{\mathrm{heq}}(Ac) = \mathrm{bdd}^{\mathrm{heq}}(A)$. Despite perhaps sounding like an intro-to-model-theory exercise, the combinatorics necessary to prove full existence for $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{a}}$ are somewhat subtle. It was recently established in [4] by Conant and the author that $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{a}}$ satisfies full existence in continuous logic and, relatedly, that $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{b}}$ satisfies full existence in discrete (and continuous) logic, answering a question of Adler [1, Quest. A.8]. While the relations of $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{a}}$ and $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{b}}$ are algebraically nice,[1] they seem to lack semantic consequences outside of certain special theories (such as those with a canonical independence relation in the sense of Adler [1, Lem. 3.2]).

While being able to build $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$-Morley sequences is certainly good, in many applications the important property is really that of being a *total* $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$-*Morley sequence*,[2] which is an A-indiscernible sequence satisfying $b_{\geq i} \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{A} b_{<i}$ for every $i < \omega$. When $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$ lacks the algebraic properties necessary to imply that all $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$-Morley sequences are total $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$-Morley sequences, it can in general be difficult to ensure their existence. Total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{a}}$-Morley sequences arise in Adler's characterization of canonical independence relations. And building total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{K}}$-Morley sequences, where $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{K}}$ is the relation of non-Kim-forking, is a crucial technical step in Kaplan and Ramsey's proofs of the symmetry of Kim-forking and the independence theorem in NSOP_1 theories [6].

In simple theories, Morley sequences over A are not generally based on A in the sense of Simon. They do however nearly satisfy this property. If I and J are Morley sequences over A with $I \equiv^{\mathrm{L}}_{A} J$,[3] then there are I' and K such that $I + I'$, $I' + K$, and $J + K$ are A-indiscernible. In an NSOP_1 theory T, if I is a tree Morley sequence over $M \models T$ and $J \equiv_M I$, then we can find K_0, K_1, and K_2 such that $I + K_0$, $K_1 + K_0$, $K_1 + K_2$, and $J + K_2$ are all M-indiscernible (see Proposition 4.28). These facts suggest the consideration of the following equivalence relation, originally introduced by Shelah in [9, Def. 5.1]: Let $\approx_A$ be the transitive, symmetric closure of the relation '$I + J$ is A-indiscernible.' The intuition is that what it means for an A-indiscernible sequence I to be 'based on A' is that there

[1]In the sense of the algebra of an independence relation, not the sense of the algebra in 'algebraic closure.'

[2]This use of the term 'total' in the context of Morley sequences was originally introduced in [6].

[3]The equivalence relation $\equiv^{\mathrm{L}}_{A}$ is the transitive closure of the relation 'there is a model $M \supseteq A$ such that $b \equiv_M c$.' If $b \equiv^{\mathrm{L}}_{A} c$, we say that b and c have the same *Lascar strong type* over A.

are few $\approx_A$-classes among the realizations of $\mathrm{tp}(I/A)$. We say that I is *based on A in the sense of Shelah* if there does not exist a sequence $(I_i)_{i<\kappa}$ (with κ large) such that $I_i \equiv_A I$ for each $i<\kappa$ and $I_i \not\approx_A I_j$ for each $i<j<\kappa$. A simple compactness argument shows that I is based on A in the sense of Shelah if and only if the set of realizations of $\mathrm{tp}(I/A)$ decomposes into a bounded number of $\approx_A$-classes. In [3, Def. 2.4],[4] Buechler used this relation to define a notion of canonical base. He focuses on $\varnothing$-indiscernible sequences and gives the following definition: A is a *canonical base* of the $\varnothing$-indiscernible sequence I if any automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$ fixes A pointwise if and only if it fixes the $\approx_\varnothing$-class of I. One difficulty with this concept, of course, is that not all indiscernible sequences have canonical bases in this sense (even in T^{eq}, e.g., [1, Ex. 3.13]).

Two of the problems we have mentioned—the lack of canonical bases for indiscernible sequences and the lack of semantic consequences of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{b}}$—can both be solved by an extremely blunt move: the introduction of ultraimaginary parameters. An *ultraimaginary* is an equivalence class of an arbitrary invariant equivalence relation (as opposed to a type-definable equivalence relation, as in the definition of hyperimaginaries). Every indiscernible sequence I trivially has an ultraimaginary canonical base in the sense of Buechler, i.e., the $\approx_\varnothing$-class of I itself.

Another appealing aspect of ultraimaginaries is that they characterize Lascar strong type in the same way that hyperimaginaries characterize Kim-Pillay strong type. An ultraimaginary $[b]_E$ is *bounded over A* if it has boundedly many conjugates under $\mathrm{Aut}(\mathbb{M}/A)$. We will write $\mathrm{bdd}^{\mathrm{u}}(A)$ for the class of ultraimaginaries bounded over A. In general, it turns out that b and c have the same Lascar strong type over A if and only if they 'have the same type over $\mathrm{bdd}^{\mathrm{u}}(A)$,' once this concept is defined precisely.

Pure analogical thinking might lead one to consider the following independence notion: $b \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A c$ if $\mathrm{bdd}^{\mathrm{u}}(Ab) \cap \mathrm{bdd}^{\mathrm{u}}(Ac) = \mathrm{bdd}^{\mathrm{u}}(A)$. This notion is implicit in a result of Wagner [13, Prop. 2.12], which we restate and expand slightly (Proposition 2.4): $b \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A c$ if and only if $\langle \mathrm{Autf}(\mathbb{M}/Ab) \cup \mathrm{Autf}(\mathbb{M}/Ac)\rangle = \mathrm{Autf}(\mathbb{M}/A)$ (where $\langle X\rangle$ is the group generated by X). This characterization is clearly semantically meaningful, and moreover it allows one to discuss $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$ without actually mentioning ultraimaginaries at all. One way to see why this equivalence works is the fact that ultraimaginaries are 'dual' to co-small sets of automorphisms; a group $G \leq \mathrm{Aut}(\mathbb{M})$ is *co-small* if there is a small model M such that $\mathrm{Aut}(\mathbb{M}/M) \leq G$. For every co-small group G, there is an ultraimaginary a_E such that $\mathrm{Aut}(\mathbb{M}/a_E) = G$ (Proposition 1.7).

As $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$ lacks finite character, *total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequences over A* seem to be correctly defined as A-indiscernible sequences $(b_i)_{i<\omega}$ with the property that for any $I+J \equiv^{\mathrm{EM}}_A b_{<\omega}$,[5] we have that $I \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A J$. The automorphism group characterization of $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$, together with its the nice algebraic properties and the malleability of indiscernible sequences, leads to a pleasing characterization of total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley

[4]This preprint is difficult to track down. The relevant ideas are developed further by Adler in [1, Sec. 3.2], which is easily available.

[5]$I \equiv^{\mathrm{EM}}_A J$ means that I and J have the same *Ehrenfeucht-Mostowski type* over A (i.e., for any increasing tuples $\bar{b} \in I$ and $\bar{c} \in J$ of the same length, $\bar{b} \equiv_A \bar{c}$). Note that I and J do not need to have the same order type.

sequences over sets of hyperimaginary parameters (Theorem 4.8), the equivalence of the following.

- $(b_i)_{i<\omega}$ is a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A.
- For some infinite I and J, we have that $I + J \equiv^{\mathrm{EM}}_A b_{<\omega}$ and $I \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A J$.
- For any I, $I \approx_A b_{<\omega}$ if and only if there is $I' \equiv^{\mathrm{L}}_A I$ such that $b_{<\omega} + I'$ is A-indiscernible.
- $b_{<\omega}$ is based on A in the sense of Shelah; i.e., $[b_{<\omega}]_{\approx_A} \in \mathrm{bdd}^{\mathrm{u}}(A)$.

The condition in the third bullet point is a natural mutual generalization of Lascar strong type and Ehrenfeucht-Mostowski type (Definition 4.5). Theorem 4.8 also tells us that when total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences exist, they act as particularly uniform witnesses of Lascar strong type (Proposition 4.3).

Of course this all leaves two critical questions: Does $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$ always satisfy full existence? And, even if it does, can we actually build total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences in any type over any set under any theory? The bluntness of ultraimaginaries leaves us without one of the most important tools in model theory, compactness. Furthermore, $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$'s lack of finite character gives us less leeway in applying the Erdős-Rado theorem to construct indiscernible sequences with certain properties; we now need to be more concerned with the particular order types of the sequences involved.

Using some of the indiscernible tree technology from [6], we are able to prove that $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$ does satisfy full existence over arbitrary sets of (hyperimaginary) parameters in arbitrary (discrete or continuous) theories (Theorem 3.6).[6] With regards to building total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences, Theorem 4.8 tells us that we don't need to worry *too* much about order types. All we need to get a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A is an A-indiscernible sequence $(b_i)_{i<\omega+\omega}$ with $b_{\geq\omega} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A b_{<\omega}$. This is fortunate because constructing ill-ordered $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences directly seems daunting. Unfortunately, $\omega + \omega$ appears to be about one ω further than we can go without a large cardinal. What we do get is this (Theorem 4.22): For any A and b in any theory T, if there is an Erdős cardinal $\kappa(\alpha)$ with $|Ab| + |T| < \kappa(\alpha)$ (for any $\alpha \geq \omega$), then there is a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence $(b_i)_{i<\omega}$ over A with $b_0 = b$. Without a large cardinal, the best we seem to be able to do (Proposition 4.17) is a half-infinite, half-arbitrary-finite approximation of a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence, which we call a *weakly total* $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$*-Morley sequence*. These sequences also serve as uniform witnesses of Lascar strong type without any set-theoretic hypotheses (Corollary 4.18).

1. Ultraimaginaries

Here we will set definitions and conventions, and we also take the opportunity to collect some basic facts about ultraimaginaries which are likely folklore, although we could not find explicit references.

Fix a theory T and a set-sized monster model $\mathbb{M} \models T$.

[6]Although this result partially supersedes a result in [4] (full existence for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{a}}$ in continuous logic and $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{b}}$ in discrete or continuous logic), the proof there gives more detailed numerical information which may be especially useful in the metric context.

Definition 1.1. An *invariant equivalence relation of arity* κ is an equivalence relation E on $\mathbb{M}^x$ (with $|x| = \kappa$) such that for any $a, b, c, d \in \mathbb{M}^x$ with $ab \equiv cd$, aEb if and only if cEd.

An *ultraimaginary of arity* κ is a pair (E, a_E) consisting of an invariant equivalence relation E (of arity κ) and an E-equivalence class a_E of some tuple $a \in \mathbb{M}^x$. By an abuse of notation, we will write a_E for the pair (E, a_E), and we may also write $[a]_E$ if necessary for notational clarity.

Given an ultraimaginary a_E, $\mathrm{Aut}(\mathbb{M}/a_E)$ is the set of automorphisms $\sigma \in \mathrm{Aut}(\mathbb{M})$ with the property that $aE(\sigma \cdot a)$. We write $\mathrm{Autf}(\mathbb{M}/a_E)$ for the group generated by $\{\sigma \in \mathrm{Aut}(\mathbb{M}/M) : M \preceq \mathbb{M},\ \mathrm{Aut}(\mathbb{M}/M) \leq \mathrm{Aut}(\mathbb{M}/a_E)\}$.

We say that b_F is *definable over* a_E if b_F is fixed by every automorphism in $\mathrm{Aut}(\mathbb{M}/a_E)$. We write $\mathrm{dcl}^{\mathrm{u}}(a_E)$ for the class of all ultraimaginaries definable over a_E. For any κ, we write $\mathrm{dcl}^{\mathrm{u}}_\kappa(a_E)$ for the set of elements of $\mathrm{dcl}^{\mathrm{u}}(a_E)$ of arity at most κ. We say that b_F and c_G are *interdefinable over* a_E if $b_F \in \mathrm{dcl}^{\mathrm{u}}(a_E c_G)$ and $c_G \in \mathrm{dcl}^{\mathrm{u}}(a_E b_F)$.

We say that b_F is *bounded over* a_E if the $\mathrm{Aut}(\mathbb{M}/a_E)$-orbit of b_F is bounded.[7] We write $\mathrm{bdd}^{\mathrm{u}}(a_E)$ for the class of all ultraimaginaries bounded over a_E. We write $\mathrm{bdd}^{\mathrm{u}}_\kappa(a_E)$ for the set of elements of $\mathrm{bdd}^{\mathrm{u}}(a_E)$ of arity at most κ. We say that b_F and c_G are *interbounded over* a_E if $b_F \in \mathrm{bdd}^{\mathrm{u}}(a_E c_G)$ and $c_G \in \mathrm{bdd}^{\mathrm{u}}(a_E b_F)$.

We write $a_E \equiv b_E$ to mean that there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$ with $\sigma \cdot a_E = b_E$. We write $b_F \equiv_{a_E} c_F$ to mean that $a_E b_F \equiv a_E c_F$ (i.e., there is $\sigma \in \mathrm{Aut}(\mathbb{M}/a_E)$ such that $\sigma \cdot b_F = c_F$).

Note that real elements, imaginaries, and hyperimaginaries can all be regarded as ultraimaginaries.

An easy counting argument shows that $\mathrm{bdd}^{\mathrm{u}}$ is a closure operator (i.e., for any a_E, b_F, and c_G, if $b_F \in \mathrm{bdd}^{\mathrm{u}}(a_E)$ and $c_G \in \mathrm{bdd}^{\mathrm{u}}(b_F)$, then $c_G \in \mathrm{bdd}^{\mathrm{u}}(a_E)$).

We will also sometimes define an invariant equivalence relation E on the realizations of a single type $p(x)$ over $\varnothing$. Equivalence classes of such can be thought of as ultraimaginaries by using the same trick that is commonly used with hyperimaginaries: Consider the invariant equivalence relation $E'(x, y)$ defined by $x = y \vee (E(x, y) \wedge x \models p \wedge y \models p)$.

For the sake of clarity, we will reserve the notation a_E for ultraimaginaries and write hyperimaginaries in the same way we write real elements. For the sake of cardinality issues, we will also take all hyperimaginaries to be quotients of countable tuples by countably type-definable equivalence relations. It is a standard fact that every hyperimaginary is interdefinable with some set of hyperimaginaries of this form.

Fact 1.2 (Shelah[8]). *Let $(b_i)_{i<\lambda}$ be a sequence of tuples with $|b_i| < \kappa$ and let A be some set of parameters. If $\lambda \geq \beth_{(2^{\kappa+|A|+|T|})^+}$, then there is an A-indiscernible sequence $(b'_i)_{i<\omega}$ such that for every $n < \omega$, there are $i_0 < \cdots < i_n < \kappa$ such that $b'_0 \dots b'_n \equiv_A b_{i_0} \dots b_{i_n}$.*

Lemma 1.3. *Let M be a model. If $a_E \in \mathrm{bdd}^{\mathrm{u}}(M)$, then $a_E \in \mathrm{dcl}^{\mathrm{u}}(M)$.*

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[7]Specifically, by Proposition 1.4, this is equivalent to b_F having at most $2^{|ab|+|T|}$ conjugates over a_E.

[8]See [12, Lem. 7.2.12] for a modern presentation of the result.

Proof. Assume that $a_E \notin \mathrm{dcl}^{\mathrm{u}}(M)$. Let $p(x)$ be a global M-invariant type extending $\mathrm{tp}(a/M)$. Assume that there are a_0 and a_1 realizing $\mathrm{tp}(a/M)$ such that $a_0 \not\!\!E a_1$. For any $i > 1$, given $a_{<i}$, let $a_i \models p\restriction Ma_{<i}$. Since $a_i a_j \equiv_M a_i a_k$ for any $j,k < i$, we must have that $a_i \not\!\!E a_j$ for any $j < i$. Since we can do this indefinitely, we have that a_E is not bounded over M. □

Proposition 1.4. *For any ultraimaginaries a_E and b_F, the following are equivalent.*

(1) $b_F \notin \mathrm{bdd}^{\mathrm{u}}(a_E)$.

(2) There is an a-indiscernible sequence $(c_i)_{i<\omega}$ such that $c_0 \equiv_{a_E} b$ and $c_i \not\!\!F c_j$ for each $i < j < \omega$.

(3) $|\mathrm{Aut}(\mathbb{M}/a_E) \cdot b_F| > 2^{|ab|+|T|}$.

Proof. (3)⇒(2). Let $(b^i_F)_{i<(2^{|ab|+|T|})^+}$ be an enumeration of $\mathrm{Aut}(\mathbb{M}/a_E) \cdot b_F$. Let $M \supseteq a$ be a model with $|M| \leq |a| + |T|$. Let x be a tuple of variables of the same length as b. There are at most $2^{|ab|+|T|}$ types in $S_x(M)$. Therefore, there must be $i < j < (2^{|ab|+|T|})^+$ such that $b^i \equiv_M b^j$. Let $p(x)$ be a global M-invariant type extending $\mathrm{tp}(b^i/M)$, and let $(c_i)_{i<\omega}$ be a Morley sequence generated by $p(x)$ over Mb^ib^j. Since $b^i \not\!\!F b^j$, we must have that $c_0 \not\!\!F b^i$. Therefore $c_i \not\!\!F c_j$ for any $i < j < \lambda$, and so $(c_i)_{i<\omega}$ is the required a-indiscernible sequence.

(2)⇒(1). Given an a-indiscernible sequence $(c_i)_{i<\omega}$ as in the statement of the proposition, we can extend it to an a-indiscernible sequence $(c_i)_{i<\lambda}$ for any λ. These sequences will still satisfy that $c_i \not\!\!F c_j$ for any $i < j < \lambda$, so b_F has an unbounded number of $\mathrm{Aut}(\mathbb{M}/a_E)$-conjugates and $b_F \notin \mathrm{bdd}^{\mathrm{u}}(a_E)$.

(1)⇒(3). This is immediate from the definition of $\mathrm{bdd}^{\mathrm{u}}(a_E)$. □

Corollary 1.5. *For any λ, $\mathrm{bdd}^{\mathrm{u}}_\lambda(a_E)$ has cardinality at most $2^{|a|+2^{\lambda+|T|}}$.*

Proof. For each $\alpha \leq \lambda$, $|S_{\alpha+\alpha}(T)| \leq 2^{\lambda+|T|}$. Since an invariant equivalence relation on α-tuples is specified by a subset of $S_{\alpha+\alpha}(T)$, this implies that for each $\alpha \leq \lambda$, there are at most $2^{2^{\lambda+|T|}}$ invariant equivalence relations on α-tuples. Therefore the total number of invariant equivalence relations on tuples of length at most λ is $\lambda \cdot 2^{2^{\lambda+|T|}} = 2^{2^{\lambda+|T|}}$. For each such F, the set $\{b_F : b_F \in \mathrm{bdd}^{\mathrm{u}}_\lambda(a_E)\}$ has cardinality at most $2^{|a|+\lambda+|T|}$ by Proposition 1.4. Finally, $2^{2^{\lambda+|T|}} \cdot 2^{|a|+\lambda+|T|} = 2^{|a|+2^{\lambda+|T|}}$. □

1.1. **Co-small groups of automorphisms.** Here we will see that ultraimaginaries are essentially the same thing as reasonable subgroups of $\mathrm{Aut}(\mathbb{M})$.

Definition 1.6. A group $G \leq \mathrm{Aut}(\mathbb{M})$ is *co-small* if there is a small model M such that $\mathrm{Aut}(\mathbb{M}/M) \leq G$.

Clearly for any ultraimaginary a_E, $\mathrm{Aut}(\mathbb{M}/a_E)$ is co-small. The converse is true as well.

Proposition 1.7. *For any co-small G, if $\mathrm{Aut}(\mathbb{M}/M) \leq G$, then there is an ultraimaginary a_E such that $G = \mathrm{Aut}(\mathbb{M}/a_E)$ where a is some enumeration of M.*

Proof. Let M be a small model witnessing that G is co-small. Consider the binary relation defined on realizations of $\mathrm{tp}(M)$ (in some fixed enumeration) defined by $E(M_0, M_1)$ if and only if there is $\sigma \in \mathrm{Aut}(\mathbb{M})$ and $\tau \in G$ such that $\sigma \cdot M = M_0$ and $\sigma\tau \cdot M = M_1$. We need to verify that E is an invariant equivalence relation. Reflexivity is obvious.

Invariance. Suppose that $E(M_0, M_1)$, as witnessed by $\sigma \in \mathrm{Aut}(\mathbb{M})$ and $\tau \in G$. Fix $\sigma' \in \mathrm{Aut}(\mathbb{M})$. We then have that $\sigma'\sigma \cdot M = \sigma' \cdot M_0$ and $\sigma'\sigma\tau \cdot M = \sigma' \cdot M_1$, whence $E(\sigma' \cdot M_0, \sigma' \cdot M_1)$.

Symmetry. If $\sigma \cdot M = M_0$ and $\sigma\tau \cdot M = M_1$ with $\sigma \in \mathrm{Aut}(\mathbb{M})$ and $\tau \in G$, then $\sigma\tau\tau^{-1} \cdot M = M_0$ and $\sigma\tau \cdot M = M_1$. We have $\sigma\tau \in \mathrm{Aut}(\mathbb{M})$ and $\tau^{-1} \in G$, so $E(M_1, M_0)$.

Transitivity. Suppose that for $\sigma, \sigma' \in \mathrm{Aut}(\mathbb{M})$ and $\tau, \tau' \in G$, we have that $\sigma \cdot M = M_0$, $\sigma\tau \cdot M = \sigma' \cdot M = M_1$, and $\sigma'\tau' \cdot M = M_2$. This implies that $(\sigma\tau)^{-1}\sigma' = \tau^{-1}\sigma^{-1}\sigma' \in \mathrm{Aut}(\mathbb{M}/M) \leq G$. Since $\tau \in G$ as well, we have that $\sigma^{-1}\sigma' \in G$. Therefore $\sigma^{-1}\sigma'\tau' \in G$. Finally, $\sigma\sigma^{-1}\sigma'\tau' \cdot M = M_2$, so $E(M_0, M_2)$.

Consider the ultraimaginary M_E. For any $\tau \in G$, we clearly have $E(M, \tau \cdot M)$, so $G \leq \mathrm{Aut}(\mathbb{M}/M_E)$. Conversely, suppose that $\alpha \in \mathrm{Aut}(\mathbb{M}/M_E)$. By definition, this implies that $E(M, \alpha \cdot M)$, so there are $\sigma \in \mathrm{Aut}(\mathbb{M})$ and $\tau \in G$ such that $\sigma \cdot M = M$ and $\sigma\tau \cdot M = \alpha \cdot M$. Therefore $\sigma, \tau^{-1}\sigma^{-1}\alpha \in \mathrm{Aut}(\mathbb{M}/M) \leq G$. Since $\tau^{-1} \in G$, we therefore have that $\alpha \in G$. □

Corollary 1.8. *If $b_F \in \mathrm{bdd}^{\mathrm{u}}(a_E)$, then there is $c_G \in \mathrm{bdd}^{\mathrm{u}}(a_E)$ of arity at most $|a| + |T|$ such that b_F and c_G are interdefinable over $\varnothing$. Furthermore, c can be taken to be an enumeration of any model of size at most $|a| + |T|$ containing a.*

Proof. There is a model $M \supseteq a$ with $|M| \leq |a| + |T|$. By Lemma 1.3, we have that $\mathrm{Aut}(\mathbb{M}/M) \leq \mathrm{Aut}(\mathbb{M}/b_F)$, so by Proposition 1.7, we have that there is c_G with arity at most $|a| + |T|$ which satisfies that $\mathrm{Aut}(\mathbb{M}/c_G) = \mathrm{Aut}(\mathbb{M}/b_F)$ (i.e., c_G and b_F are interdefinable over $\varnothing$). Furthermore, we can take c to be an enumeration of M. □

Definition 1.9. For any co-small group G, we write $[[G]]$ for some arbitrary ultraimaginary a_E of minimal arity satisfying $G = \mathrm{Aut}(\mathbb{M}/a_E)$. We will write $\mathrm{dcl}^{\mathrm{u}}[[G]]$ and $\mathrm{dcl}^{\mathrm{u}}_\lambda[[G]]$ for $\mathrm{dcl}^{\mathrm{u}}([[G]])$ and $\mathrm{dcl}^{\mathrm{u}}_\lambda([[G]])$ and likewise with $\mathrm{bdd}^{\mathrm{u}}$. (Note that $\mathrm{dcl}^{\mathrm{u}}[[G]]$ and $\mathrm{bdd}^{\mathrm{u}}[[G]]$ only depend on G, not on the particular choice of $[[G]]$.)

It is immediate from Proposition 1.7 that for any co-small G and H, $[[G]] \in \mathrm{dcl}^{\mathrm{u}}[[H]]$ if and only if $G \geq H$. A similar statement for $\mathrm{bdd}^{\mathrm{u}}$ is given in Proposition 1.12 below.

Now we can see that intersections of $\mathrm{dcl}^{\mathrm{u}}$-closed sets (and therefore also $\mathrm{bdd}^{\mathrm{u}}$-closed sets) have semantic significance in arbitrary theories, in that intersections correspond to joins in the lattice of co-small groups of automorphisms.

Proposition 1.10. *For any a_E, b_F, c_G, and c'_G, the following are equivalent.*

(1) $c_G \equiv_{\mathrm{dcl}^{\mathrm{u}}_\lambda(a_E) \cap \mathrm{dcl}^{\mathrm{u}}_\lambda(b_F)} c'_G$ *for all* λ.
(2) *There is* $\sigma \in \langle \mathrm{Aut}(\mathbb{M}/a_E) \cup \mathrm{Aut}(\mathbb{M}/b_F) \rangle$ *such that* $\sigma \cdot c_G = c'_G$.
(3) *There is a sequence* $(a^i b^i c^i)_{i \leq n}$ *such that* $a^0 = a$, $b^0 = b$, $c^0 = c$, $c^n_G = c'_G$, *and for each* $i < n$,
   - *if i is even, then* $a^i = a^{i+1}$ *and* $b^i_F c^i_G \equiv_{a^i_E} b^{i+1}_F c^{i+1}_G$ *and*
   - *if i is odd, then* $b^i = b^{i+1}$ *and* $a^i_E c^i_G \equiv_{b^i_F} a^{i+1}_E c^{i+1}_G$.

Proof. Let $H = \langle \mathrm{Aut}(\mathbb{M}/a_E) \cup \mathrm{Aut}(\mathbb{M}/b_F) \rangle$.

Claim. $\mathrm{dcl}^{\mathrm{u}}_\lambda(a_E) \cap \mathrm{dcl}^{\mathrm{u}}_\lambda(b_F)$ and $[[H]]$ are interdefinable (i.e., $\mathrm{dcl}^{\mathrm{u}}_\lambda(a_E) \cap \mathrm{dcl}^{\mathrm{u}}_\lambda(b_F) \subseteq \mathrm{dcl}^{\mathrm{u}}([[H]])$ and $[[H]] \in \mathrm{dcl}^{\mathrm{u}}(\mathrm{dcl}^{\mathrm{u}}_\lambda(a_E) \cap \mathrm{dcl}^{\mathrm{u}}_\lambda(b_F))$) for all sufficiently large λ.

Proof of claim. Clearly $[[H]] \in \mathrm{dcl}^{\mathrm{u}}(a_E) \cap \mathrm{dcl}^{\mathrm{u}}(b_F)$, so $[[H]] \in \mathrm{dcl}^{\mathrm{u}}_\lambda(a_E) \cap \mathrm{dcl}^{\mathrm{u}}_\lambda(b_F)$ for all sufficiently large λ.

Conversely, suppose that $d_I \in \mathrm{dcl}^{\mathrm{u}}(a_E) \cap \mathrm{dcl}^{\mathrm{u}}(b_F)$. Any $\sigma \in H$ is a product of elements of $\mathrm{Aut}(\mathbb{M}/a_E)$ and $\mathrm{Aut}(\mathbb{M}/b_F)$, so it must fix d_I. Therefore $\mathrm{Aut}(\mathbb{M}/d_I) \geq H$ and hence $d_I \in \mathrm{dcl}^{\mathrm{u}}[[H]]$. $\square_{\text{claim}}$

So now we have that $c_G \equiv_{\mathrm{dcl}^{\mathrm{u}}_\lambda(a_E)\cap \mathrm{dcl}^{\mathrm{u}}_\lambda(b_F)} c'_G$ holds for sufficiently large λ if and only if $c_G \equiv_{[[H]]} c'_G$. Also note that $c_G \equiv_{\mathrm{dcl}^{\mathrm{u}}_\lambda(a_E)\cap \mathrm{dcl}^{\mathrm{u}}_\lambda(b_F)} c'_G$ for sufficiently large λ and only if the same holds for any λ. Therefore we have that (1) and (2) are equivalent.

There is a $\sigma \in H$ with $\sigma \cdot c_G = c'_G$ if and only if there are $\alpha_0, \dots, \alpha_{n-1} \in \mathrm{Aut}(\mathbb{M}/a_E)$ and $\beta_0, \dots, \beta_{n-1} \in \mathrm{Aut}(\mathbb{M}/b_F)$ such that $\sigma = \alpha_{n-1}\beta_{n-1}\dots\beta_1\alpha_0\beta_0$.

For (2)⇒(3), assume that there are such $\bar{\alpha}$ and $\bar{\beta}$ for which

$$\alpha_{n-1}\beta_{n-1}\alpha_{n-2}\dots\beta_1\alpha_0\beta_0 \cdot c_G = c'_G.$$

Let $a^0b^0c^0 = abc$, $a^1b^1c^1 = \alpha_{n-1} \cdot (a^0b^0c^0)$, $a^2b^2c^2 = \alpha_{n-1}\beta_{n-1} \cdot (a^0b^0c^0)$, and so on up to $a^{2n}b^{2n}c^{2n} = \alpha_{n-1}\beta_{n-1}\alpha_{n-2}\dots\beta_1\alpha_0\beta_0 \cdot (a^0b^0c^0)$. Clearly we have that $c^{2n}_G = c'_G$, so we just need to verify that $(a^ib^ic^i)_{i\leq 2n}$ is the required sequence. If $i < 2n$ is even, then $\alpha_i \in \mathrm{Aut}(\mathbb{M}/a_E)$, so $a^i_E = a^{i+1}_E$. Furthermore, $b^0_Fc^0_G \equiv_{a^0_E} \alpha_i \cdot (b^0_Fc^0_G)$, so by invariance,

$$\alpha_{n-1}\beta_{n-1}\dots\beta_{i+1} \cdot (b^0_Fc^0_G) \equiv_{\alpha_{n-1}\beta_{n-1}\dots\beta_{i+1}\cdot a^0_E} \alpha_{n-1}\beta_{n-1}\dots\beta_{i+1}\alpha_i \cdot (b^0_Fc^0_G),$$

which is the same as $b^i_Fc^i_G \equiv_{a^i_E} b^{i+1}_Fc^{i+1}_G$. If $i < 2n$ is odd, then the same argument tells us that $b^i_F = b^{i+1}_F$ and $a^i_Ec^i_G \equiv_{b^i_F} a^{i+1}_Ec^{i+1}_G$.

For (3)⇒(2), the above argument is reversible. Fix $(a^i_Eb^i_Fc^i_G)_{i\leq 2n}$ satisfying the conditions of (3). First we can find $\alpha_{n-1} \in \mathrm{Aut}(\mathbb{M}/a_E)$ such that $\alpha^{-1}_{n-1}\cdot(a^1_Eb^1_Fc^1_G) = a^0_Eb^0_Fc^0_G$. Then we can find $\beta_{n-1} \in \mathrm{Aut}(\mathbb{M}/b_F)$ such that $\beta^{-1}_{n-1}\alpha^{-1}_{n-1} \cdot (a^2_Eb^2_Fc^2_G) = a^0_Eb^0_Fc^0_G$. Then we can find $\alpha_{n-2} \in \mathrm{Aut}(\mathbb{M}/a_E)$ such that $\alpha^{-1}_{n-2}\beta^{-1}_{n-1}\alpha^{-1}_{n-1}\cdot(a^3_Eb^3_Fc^3_G) = a^0_Eb^0_Fc^0_G$. Continuing inductively in this way, we find $\alpha_0, \dots, \alpha_{n-1} \in \mathrm{Aut}(\mathbb{M}/a_E)$ and $\beta_0, \dots, \beta_{n-1} \in \mathrm{Aut}(\mathbb{M}/b_F)$ such that the same equalities as in the (2)⇒(3) proof hold. Therefore there is a $\sigma \in H$ (namely $\alpha_{n-1}\beta_{n-1}\alpha_{n-2}\dots\beta_1\alpha_0\beta_0$) such that $\sigma \cdot c_G = c'_G$. $\square$

A similar statement is true for arbitrary families of ultraimaginaries: If $(a^i_{E_i})_{i\in I}$ is a (possibly large) family of ultraimaginaries, then $c_G \equiv_{\bigcap_{i\in I}\mathrm{dcl}^{\mathrm{u}}_\lambda(a^i_{E_i})} c'_G$ if and only if there is a $\sigma \in \left\langle \bigcup_{i\in I}\mathrm{Aut}(\mathbb{M}/a^i_{E_i})\right\rangle$ such that $\sigma \cdot c_G = c'_G$. There is also an analog of (3), but it is more awkward to state.

1.2. **Lascar strong type.**

Definition 1.11. For any co-small group $G \leq \mathrm{Aut}(\mathbb{M})$, let G_f be the group generated by all groups of the form $\mathrm{Aut}(\mathbb{M}/M) \leq G$ with M a small model. For any ultraimaginary a_E, let $\mathrm{Autf}(\mathbb{M}/a_E) = \mathrm{Aut}(\mathbb{M}/a_E)_f$.

We say that b_F and c_F have the same *Lascar strong type over* a_E, written $b_F \equiv^{\mathrm{L}}_{a_E} c_F$, if there is $\sigma \in \mathrm{Autf}(\mathbb{M}/a_E)$ such that $\sigma \cdot b_F = c_F$.

Proposition 1.12. *For any co-small groups G and H, $[[G]] \in \mathrm{bdd}^{\mathrm{u}}[[H]]$ if and only if $G \geq H_f$.*

Proof. Assume that $[[G]] \in \mathrm{bdd}^{\mathrm{u}}[[H]]$. Note that for a model M, by Lemma 1.3, we have that $[[G]] \in \mathrm{bdd}^{\mathrm{u}}(M)$ if and only if $G \geq \mathrm{Aut}(\mathbb{M}/M)$. Therefore, for any model M with $[[H]] \in \mathrm{bdd}^{\mathrm{u}}(M)$, we must have that $[[G]] \in \mathrm{bdd}^{\mathrm{u}}[[H]] \subseteq \mathrm{bdd}^{\mathrm{u}}(M)$ and so $G \geq \mathrm{Aut}(\mathbb{M}/M)$. Since $[[H]] \in \mathrm{bdd}^{\mathrm{u}}(M)$ if and only if $H \geq \mathrm{Aut}(\mathbb{M}/M)$, we have that $G \geq H_f$.

Conversely, assume that $G \geq H_f$. This implies that for any small model M with $[[H]] \in \mathrm{bdd}^{\mathrm{u}}(M)$, we have $H_f \geq \mathrm{Aut}(\mathbb{M}/M)$, so $G \geq \mathrm{Aut}(\mathbb{M}/M)$ and $[[G]] \in \mathrm{dcl}^{\mathrm{u}}(M)$. Fix some such model N. Assume for the sake of contradiction that $[[G]] \notin \mathrm{bdd}^{\mathrm{u}}[[H]]$. For any λ, we can find $(\sigma_i)_{i<\lambda}$ in $H = \mathrm{Aut}(\mathbb{M}/[[H]])$ such that $\sigma_i \cdot [[G]] \neq \sigma_j \cdot [[G]]$ for each $i < j < \lambda$. Since $[[G]] = a_E$ for some a with $|a| \leq |N|$ by Proposition 1.7, we have that if λ is larger than $2^{|N|+|T|}$, there must be $i < j < \lambda$ such that $\sigma_i \cdot [[G]] \equiv_N \sigma_j \cdot [[G]]$. Let $N' = \sigma_i^{-1} \cdot N$. N' is now a model satisfying $\mathrm{Aut}(\mathbb{M}/N') \leq G$. So $[[G]] \in \mathrm{dcl}^{\mathrm{u}}(N')$, but $[[G]] \equiv_{N'} \sigma_i^{-1}\sigma_j \cdot [[G]]$ and $[[G]] \neq \sigma_i^{-1}\sigma_j \cdot [[G]]$, which is a contradiction. □

An important fact about ultraimaginaries is that $\mathrm{bdd}^{\mathrm{u}}$ has the same relationship with Lascar strong types that $\mathrm{bdd}^{\mathrm{heq}}$ has with Kim-Pillay strong types.

For any a_E and b_F, by an abuse of notation, we'll write $[b_F]_{\equiv^{\mathrm{L}}_{a_E}}$ for $[ab]_G$, where $G(ab, a'b')$ holds if and only if aEa' and $b_F \equiv^{\mathrm{L}}_{a_E} b'_F$. Note in particular that $[b_F]_{\equiv^{\mathrm{L}}_{a_E}} = [b'_F]_{\equiv^{\mathrm{L}}_{a_E}}$ if and only if $b_F \equiv^{\mathrm{L}}_{a_E} b'_F$.

Proposition 1.13. *For any ultraimaginaries a_E, b_F, and c_F, the following are equivalent.*

(1) $b_F \equiv^{\mathrm{L}}_{a_E} c_F$.
(2) $b_F \equiv_{\mathrm{bdd}^{\mathrm{u}}_\lambda(a_E)} c_F$ *for all* λ.
(3) $b_F \equiv_{\mathrm{bdd}^{\mathrm{u}}_{|a|+|T|}(a_E)} c_F$.

Proof. To see that (1) implies (3), fix a model M with $a_E \in \mathrm{bdd}^u(M)$ and some automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$. By Lemma 1.3, we have that $\mathrm{Aut}(\mathbb{M}/M) \leq \mathrm{Aut}(\mathbb{M}/\mathrm{bdd}^{\mathrm{u}}_{|a|+|T|}(a_E))$. Therefore $b_F \equiv_{\mathrm{bdd}^{\mathrm{u}}_{|a|+|T|}(a_E)} \sigma \cdot b_F$. By induction, we therefore have that $b_F \equiv^{\mathrm{L}}_{a_E} c_F$ implies $b_F \equiv_{\mathrm{bdd}^{\mathrm{u}}_{|a|+|T|}(a_E)} c_F$.

Corollary 1.8 implies that $\mathrm{Aut}(\mathbb{M}/\mathrm{bdd}^{\mathrm{u}}_\lambda(a_E)) \geq \mathrm{Aut}(\mathbb{M}/\mathrm{bdd}^{\mathrm{u}}_{|a|+|T|}(a_E))$ for all λ, so (3) implies (2).

To see that (2) implies (1), note that $[b_F]_{\equiv^{\mathrm{L}}_{a_E}} \in \mathrm{bdd}^{\mathrm{u}}_\lambda(a_E)$ for some sufficiently large λ (because there are a bounded number of Lascar strong types over a_E). Therefore if $b_F \equiv_{\mathrm{bdd}^{\mathrm{u}}_\lambda(a_E)} c_F$, we must have $[b_F]_{\equiv^{\mathrm{L}}_{a_E}} = [c_F]_{\equiv^{\mathrm{L}}_{a_E}}$ or, in other words, $b_F \equiv^{\mathrm{L}}_{a_E} c_F$. □

2. Bounded ultraimaginary independence

Definition 2.1. Given sets of ultraimaginaries A, B, and C, we write $B \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A C$ to mean that $\mathrm{bdd}^{\mathrm{u}}(AB) \cap \mathrm{bdd}^{\mathrm{u}}(AC) = \mathrm{bdd}^{\mathrm{u}}(A)$.

Recall that $\mathrm{bdd}^{\mathrm{u}}$ is a closure operator (i.e., if $c_G \in \mathrm{bdd}^{\mathrm{u}}(b_F)$ and $b_F \in \mathrm{bdd}^{\mathrm{u}}(a_E)$, then $c_G \in \mathrm{bdd}^{\mathrm{u}}(a_E)$). We will ultimately show (in Proposition 2.3) that the following are equivalent: $b_F \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_{a_E} c_G$, $\mathrm{bdd}^{\mathrm{u}}_\kappa(a_E b_F) \cap \mathrm{bdd}^{\mathrm{u}}_\kappa(a_E c_G) = \mathrm{bdd}^{\mathrm{u}}_\kappa(a_E)$ for all κ, and $\mathrm{bdd}^{\mathrm{u}}_\kappa(a_E b_F) \cap \mathrm{bdd}^{\mathrm{u}}_\kappa(a_E c_G) = \mathrm{bdd}^{\mathrm{u}}_\kappa(a_E)$ for $\kappa = |T| + |abc|$. $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$ satisfies some of the familiar properties of $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{a}}$.

Proposition 2.2. *Fix ultraimaginaries a_E, b_F, c_G, and e_I.*

- (Invariance) *If $a_E b_F c_G \equiv a'_E b'_F c'_G$, then $b_F \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_{a_E} c_G$ if and only if $b'_F \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_{a'_E} c'_G$.*

- (Symmetry) $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$ *if and only if* $c_G \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} b_F$.
- (Monotonicity) *If* $b_F c_G \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} d_H e_I$, *then* $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} d_H$.
- (Transitivity) *If* $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$ *and* $d_H \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E b_F} c_G$, *then* $b_F d_H \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$.
- (Normality) *If* $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$, *then* $a_E b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} a_E c_G$.
- (Anti-reflexivity) *If* $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} b_F$, *then* $b_F \in \operatorname{bdd}^{\text{u}}(a_E)$.

Proof. Everything except transitivity is immediate. The argument for transitivity is the same as the argument for transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{\text{a}}$: Assume that $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$ and $d_H \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E b_F} c_G$. Let e_I be an element of $\operatorname{bdd}^{\text{u}}(a_E b_F d_H) \cap \operatorname{bdd}^{\text{u}}(a_E c_G)$. This implies that it is an element of $\operatorname{bdd}^{\text{u}}(a_E b_F d_H) \cap \operatorname{bdd}^{\text{u}}(a_E b_F c_G)$, so by assumption it is an element of $\operatorname{bdd}^{\text{u}}(a_E b_F)$. But this means that it's in both $\operatorname{bdd}^{\text{u}}(a_E b_F)$ and $\operatorname{bdd}^{\text{u}}(a_E c_G)$, so, by assumption again, it is an element of $\operatorname{bdd}^{\text{u}}(a_E)$. □

Part of the goal of this paper is to prove full existence and therefore also extension for $\mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}$ (although only over hyperimaginary bases).

- (Full existence over hyperimaginaries) For any set of hyperimaginaries A and ultraimaginaries b_E and c_F, there is $c'_F \equiv_A c_F$ such that $b_E \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{A} c'_F$.
- (Extension over hyperimaginaries) For any set of hyperimaginaries A and ultraimaginaries b_E, c_F, and d_G, if $b_E \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{A} c_F$, then there is $b'_E \equiv_{Ac_F} b_E$ such that $b'_E \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{A} c_F d_G$.

A fairly general argument will allow us to upgrade $\equiv_A$ to $\equiv^{\text{L}}_A$ in the above two conditions, which we establish in Theorem 3.6 and Corollary 3.8.

Finite character fails very badly, of course: As considered in [13, Ex. 2.8], if E is the equivalence relation on ω-tuples of equality on cofinitely many indices, then for some sequences $(a_i)_{i<\omega}$, we will have $a_{<n} \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}} [a_{<\omega}]_E$ for all n, yet $a_{<\omega} \not\mathop{\smile\!\!\!\!\!\!|}^{\text{bu}} [a_{<\omega}]_E$. Given the existence of higher and higher cardinality generalizations of the previous example (e.g., equality on co-countably many indices on ω_1-tuples), local character seems unlikely except possibly in the presence of large cardinals. We do have some control over the relevant cardinalities, however.

Proposition 2.3. *For any* a_E, b_F, *and* c_G, $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$ *if and only if* $\operatorname{bdd}^{\text{u}}_\lambda(a_E b_F) \cap \operatorname{bdd}^{\text{u}}_\lambda(a_E c_G) = \operatorname{bdd}^{\text{u}}_\lambda(a_E)$, *where* $\lambda = |ab| + |T|$.

Proof. Let $\lambda = |ab| + |T|$. Clearly we have that if $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$, then $\operatorname{bdd}^{\text{u}}_\lambda(a_E b_F) \cap \operatorname{bdd}^{\text{u}}_\lambda(a_E c_G) = \operatorname{bdd}^{\text{u}}_\lambda(a_E)$.

Conversely, assume that $b_F \not\mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$. There is some $d_H \in (\operatorname{bdd}^{\text{u}}(a_E b_F) \cap \operatorname{bdd}^{\text{u}}(a_E c_G)) \setminus \operatorname{bdd}^{\text{u}}(a_E)$. By Corollary 1.8, there is e_I of arity at most λ such that d_H and e_I are interdefinable. This means that $e_I \in (\operatorname{bdd}^{\text{u}}_\lambda(a_E b_F) \cap \operatorname{bdd}^{\text{u}}_\lambda(a_E c_G)) \setminus \operatorname{bdd}^{\text{u}}_\lambda(a_E)$. Therefore $\operatorname{bdd}^{\text{u}}_\lambda(a_E b_F) \cap \operatorname{bdd}^{\text{u}}_\lambda(a_E c_G) \neq \operatorname{bdd}^{\text{u}}_\lambda(a_E)$. □

The following characterization of $\mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}$ (and the manner of proof) is essentially due to Wagner [13].

Proposition 2.4. *For any ultraimaginaries* a_E, b_F, *and* c_G, *the following are equivalent.*

(1) $b_F \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_{a_E} c_G$.

(2) *For any* $b'_F \equiv^{\mathrm{L}}_{a_E} b_F$*, there are* $b^0, c^0, b^1, c^1, \ldots, c^{n-1}, b^n$ *such that* $b^0 = b$, $c^0 = c$, $b^n = b'$*, and for each* $i < n$, $b^i_F \equiv^{\mathrm{L}}_{a_E c^i_G} b^{i+1}_F$ *and* $c^i_G \equiv^{\mathrm{L}}_{a_E b^{i+1}_F} c^{i+1}_G$ *if* $i < n-1$.

(3) $\langle \mathrm{Autf}(\mathbb{M}/a_E b_F) \cup \mathrm{Autf}(\mathbb{M}/a_E c_G)\rangle = \mathrm{Autf}(\mathbb{M}/a_E)$.

Proof. Let $H = \langle \mathrm{Autf}(\mathbb{M}/a_E b_F) \cup \mathrm{Autf}(\mathbb{M}/a_E c_G)\rangle$.

$\neg$(3)$\Rightarrow\neg$(1). Assume that $H \neq \mathrm{Autf}(\mathbb{M}/a_E)$, which implies that $H < \mathrm{Autf}(\mathbb{M}/a_E) = \mathrm{Autf}(\mathbb{M}/a_E)_f$. By Proposition 1.12, we have that $[[H]] \notin \mathrm{bdd}^{\mathrm{u}}[[\mathrm{Autf}(\mathbb{M}/a_E)]] = \mathrm{bdd}^{\mathrm{u}}(a_E)$. But since $\mathrm{Autf}(\mathbb{M}/a_E b_F) = \mathrm{Autf}(\mathbb{M}/a_E b_F)_f \leq H$ and $\mathrm{Autf}(\mathbb{M}/a_E c_G) = \mathrm{Autf}(\mathbb{M}/a_E c_G)_f \leq H$, we have that $[[H]] \in \mathrm{bdd}^{\mathrm{u}}(a_E b_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c_G)$ again by Proposition 1.12.

(3)$\Rightarrow$(1). Suppose $H = \mathrm{Autf}(\mathbb{M}/a_E)$. Fix an ultraimaginary $d_I \in \mathrm{bdd}^{\mathrm{u}}(a_E b_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c_G)$. By Proposition 1.12, we have that $H \leq \mathrm{Autf}(\mathbb{M}/a_E d_I) \leq \mathrm{Autf}(\mathbb{M}/a_E)$, which implies that $\mathrm{Autf}(\mathbb{M}/a_E d_I) = H$. Therefore by Proposition 1.12, $d_I \in \mathrm{bdd}^{\mathrm{u}}(a_E)$. Since we can do this for any such ultraimaginary, we have that $b_F \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_{a_E} c_G$.

(1)$\Rightarrow$(2). Let $b^*_{F^*} = [[\mathrm{Autf}(\mathbb{M}/a_E b_F)]]$ and $c^*_{G^*} = [[\mathrm{Autf}(\mathbb{M}/a_E c_G)]]$. Note that $\mathrm{bdd}^{\mathrm{u}}(a_E b_F) = \mathrm{dcl}^{\mathrm{u}}(b^*_{F^*})$ and $\mathrm{bdd}^{\mathrm{u}}(a_E c_G) = \mathrm{dcl}^{\mathrm{u}}(c^*_{G^*})$ (by Definition 1.9 and Proposition 1.12). In particular, we have that $\mathrm{dcl}^{\mathrm{u}}(b^*_{F^*}) \cap \mathrm{dcl}^{\mathrm{u}}(c^*_{G^*}) = \mathrm{bdd}^{\mathrm{u}}(a_E)$. Fix $b'_F \equiv^{\mathrm{L}}_{a_E} b_F$. By passing to a different representative of the F-equivalence class b'_F, we may assume that $b' \equiv^{\mathrm{L}}_{a_E} b$. Fix c' such that $bc \equiv^{\mathrm{L}}_{a_E} b'c'$. By Proposition 1.13, we have that $b'c' \equiv_{\mathrm{bdd}^{\mathrm{u}}_\lambda(a_E)} bc$ for all λ, so $b'c' \equiv_{\mathrm{dcl}^{\mathrm{u}}_\lambda(b^*_{F^*}) \cap \mathrm{dcl}^{\mathrm{u}}_\lambda(c^*_{G^*})} bc$ for all λ. Therefore, by Proposition 1.10, we can find a sequence $(b^{*i}c^{*i}b^i c^i)_{i \leq n}$ such that $b^{*0} = b^*$, $c^{*0} = c^*$, $b^0 c^0 = bc$, $b^n c^n = b'c'$, and for each $i < n$,

- if i is even, $b^{*i} = b^{*i+1}$ and $c^{*i} b^i c^i \equiv_{b^{*i}} c^{*i+1} b^{i+1} c^{i+1}$ and
- if i is odd, $c^{*i} = c^{*i+1}$ and $b^{*i} b^i c^i \equiv_{c^{*i}} b^{*i+1} b^{i+1} c^{i+1}$.

This implies, by induction, that $b^i c^i \equiv^{\mathrm{L}}_{a_E b^i_F} b^{i+1} c^{i+1}$ and $b^i_F = b^{i+1}_F$ for each even i and $b^i c^i \equiv^{\mathrm{L}}_{a_E c^i_G} b^{i+1} c^{i+1}$ and $c^i_G = c^{i+1}_G$ for each odd i, so $b^0 c^1 b^2 c^3 \ldots c^{n-1} b^n$ is the sequence required by the proposition (after reindexing).

(2)$\Rightarrow$(1). Assume (2), but also assume for the sake of contradiction that (1) fails. Let d_H be an element of $(\mathrm{bdd}^{\mathrm{u}}(a_E b_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c_G)) \setminus \mathrm{bdd}^{\mathrm{u}}(a_E)$. Since d_H is not bounded over a_E, there must be some $d'_H \equiv^{\mathrm{L}}_{a_E} d_H$ such that $d'_H \notin \mathrm{bdd}^{\mathrm{u}}(a_E b_E) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c_G)$. Find b'_F such that $b_F d_H \equiv^{\mathrm{L}}_{a_E} b'_F d'_H$. Let $b^0, c^0, b^1, c^1 \ldots, c^{n-1}, b^n$ be as in (2), with $b^n = b'$. Find $d^{1/2}, d^1, d^{3/2}, d^2, \ldots, d^{n-1/2}, d^n$ such that $d^{1/2} = d$ and for each $i < n$,

- $b^i_F d^{i+1/2}_H \equiv^{\mathrm{L}}_{a_E c^i_G} b^{i+1}_F d^{i+1}$ and
- $c^i_G d^{i+1}_H \equiv^{\mathrm{L}}_{a_E b^{i+1}_F} c^{i+1}_G d^{i+3/2}_H$ if $i < n-1$.

We now have that $b'_F d'_H \equiv^{\mathrm{L}}_{a_E} b_F d_H \equiv^{\mathrm{L}}_{a_E} b'_F d^n_H$, so in particular, $d'_H \equiv^{\mathrm{L}}_{a_E b'_F} d^n_H$. For some $i < n$, consider $e_I \in \mathrm{bdd}^{\mathrm{u}}(a_E b^i_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c^i_G)$. Since $e_I \in \mathrm{bdd}^{\mathrm{u}}(a_E c^i_G)$ and since $b^i_F d^{i+1/2}_H \equiv_{\mathrm{bdd}^{\mathrm{u}}_\lambda(a_E c^i_G)} b^{i+1}_F d^{i+1}$ for all λ (by Proposition 1.13), we must have that $b^i_F d^{i+1/2}_H \equiv_{a_E e_I} b^{i+1}_F d^{i+1}$ and so $e_I \in \mathrm{bdd}^{\mathrm{u}}(a_E b^{i+1}_F)$ as well. By the reverse argument and since we can do this for any such ultraimaginary, we get that

$$\mathrm{bdd}^{\mathrm{u}}(a_E b^i_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c^i_G) = \mathrm{bdd}^{\mathrm{u}}(a_E b^{i+1}_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c^i_G).$$

Likewise, for any $i < n-1$, we get

$$\mathrm{bdd}^{\mathrm{u}}(a_E b^{i+1}_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c^i_G) = \mathrm{bdd}^{\mathrm{u}}(a_E b^{i+1}_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c^{i+1}_G).$$

Therefore $d_H^n \in \mathrm{bdd}^{\mathrm{u}}(a_E b_F^n) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c_G^{n-1})$, so since $d_H^n \equiv^{\mathrm{L}}_{a_E b_F^n} d'_H$ and so $d_H^n \equiv_{\mathrm{bdd}^{\mathrm{u}}_\lambda(a_E b_F^n)} d'_H$ for every λ (by Proposition 1.13), we must also have $d'_H \in \mathrm{bdd}^{\mathrm{u}}(a_E b_F^n) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c_G^{n-1}) = \mathrm{bdd}^{\mathrm{u}}(a_E b_F) \cap \mathrm{bdd}^{\mathrm{u}}(a_E c_G)$, which is a contradiction. □

3. Full existence

We will use the tree bookkeeping machinery from [6], with some minor extensions (the notation $\mathcal{T}_\alpha^*$ and $\mathcal{F}_\alpha$).

Definition 3.1. For any ordinal α, $\mathcal{L}_{s,\alpha}$ is the language

$$\{\trianglelefteq, \wedge, <_{\mathrm{lex}}, P_0, P_1, \dots, P_\beta(\beta < \alpha), \dots\},$$

with $\trianglelefteq$ and $<_{\mathrm{lex}}$ binary relations, $\wedge$ a binary function, and each P_β a unary relation.

For any ordinal α, we write $\mathcal{T}_\alpha^*$ for the set of functions f with codomain ω and finite support such that $\mathrm{dom}(f)$ is an end segment of α. (For the sake of some minor edge cases, we will regard the empty functions in various $\mathcal{T}_\alpha^*$'s as distinct objects.) We write $\mathcal{T}_\alpha$ for the set of functions $f \in \mathcal{T}_\alpha^*$ with $\mathrm{dom}(f) = [\beta, \alpha)$ for a non-limit ordinal β. We write $\mathcal{F}_{\alpha+1}$ (for forest) for $\mathcal{T}_{\alpha+1} \setminus \{\varnothing\}$.

We interpret $\mathcal{T}_\alpha^*$ and $\mathcal{T}_\alpha$ as $\mathcal{L}_{s,\alpha}$-structures by

- $f \trianglelefteq g$ if and only if $f \subseteq g$;
- $f \wedge g = f\restriction[\beta, \alpha) = g\restriction[\beta, \alpha)$, where $\beta = \min\{\gamma : f\restriction[\gamma, \alpha) = g\restriction[\gamma, \alpha)\}$ (with the understanding that $\min \varnothing = \alpha$);
- $f <_{\mathrm{lex}} g$ if and only if either $f \triangleleft g$ or f and g are $\trianglelefteq$-incomparable, $\mathrm{dom}(f \wedge g) = [\gamma, \alpha)$, and $f(\gamma) < g(\gamma)$; and
- $P_\beta(f)$ holds if and only if $\mathrm{dom}(f) = [\beta, \alpha)$.

We write $\langle i \rangle_\alpha$ for the function $\{(\alpha, i)\}$ (which is an element of $\mathcal{T}_{\alpha+1}^*$). Given $i < \omega$ and $f \in \mathcal{T}_\alpha^*$ with $\mathrm{dom}(f) = [\beta + 1, \alpha)$, we write $f \frown i$ to mean the function $f \cup \{(\beta, i)\}$ (which is an element of $\mathcal{T}_\alpha^*$). Given $i < \omega$ and $f \in \mathcal{T}_\alpha^*$, we write $i \frown f$ to mean the function $\{(\alpha, i)\} \cup f$ (which is an element of $\mathcal{T}_{\alpha+1}^*$).[9]

For $\alpha < \beta$, we define the canonical inclusion map $\iota_{\alpha\beta} : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ by $\iota_{\alpha\beta}(f) = f \cup \{(\gamma, 0) : \gamma \in \beta \setminus \alpha\}$. (Note that $\iota_{\alpha,\alpha+1}(f) = 0 \frown f$.)

For $\beta \leq \alpha$, we write ζ_β^α for the function whose domain is $[\beta, \alpha)$ with the property that $\zeta_\beta^\alpha(\gamma) = 0$ for all $\gamma \in [\beta, \alpha)$. (Note that ζ_α^α is $\mathcal{T}_\alpha$'s copy of the empty function.)

Given a family $(b_f)_{f \in X}$, we may refer to it briefly as $b_{\in X}$.

Definition 3.2. Given $X \subseteq \mathcal{T}_\alpha^*$, we say that a family $(b_f)_{f \in X}$ is *s-indiscernible over* A if for any tuples $f_0 \dots f_{n-1}$ and $g_0 \dots g_{n-1}$ in X with $f_0 \dots f_{n-1} \equiv^{\mathrm{qf}} g_0 \dots g_{n-1}$, $b_{f_0} \dots b_{f_{n-1}} \equiv_A b_{g_0} \dots b_{g_{n-1}}$, where quantifier-free type is in the language $\mathcal{L}_{s,\alpha}$. (Note that this does not entail that b_f's on different levels are tuples of the same sort.)

Given $f \in \mathcal{T}_\alpha$, we write $b_{\trianglerighteq f}$ to refer to some fixed enumeration of the set $\{b_g : g \in \mathcal{T}_\alpha,\ f \trianglelefteq g\}$. In particular, we choose this enumeration in a uniform way so that if $(b_f)_{f \in \mathcal{T}_\alpha}$ is s-indiscernible over A, then for any f with domain $[\beta + 1, \alpha)$, the sequence $(b_{\trianglerighteq f \frown i})_{i < \omega}$ is A-indiscernible. When f is an element of $\mathcal{T}_\alpha^*$, we will also write $b_{\trianglerighteq f}$ for some fixed enumeration of the set $\{b_g : g \in \mathcal{T}_\alpha,\ f \subseteq g\}$. One

[9] Note that this notation is not ambiguous when f is an empty function, as we are regarding the empty functions in different $\mathcal{T}_\alpha^*$'s as distinct objects.

particular example of this will be sequences of the form $(b_{\trianglerighteq\zeta^{\alpha}_{\beta+1}\frown i})_{i<\omega}$, where β is a limit ordinal. This is essentially the only situation in which we need to consider $\mathcal{T}^*_\alpha$.

Note that for a limit ordinal α, $(b_f)_{f\in\mathcal{T}_\alpha}$ is s-indiscernible over A if and only if $(b_f)_{f\in\iota_{\beta,\alpha}(\mathcal{T}_\beta)}$ is s-indiscernible over A for every $\beta<\alpha$.

We will also need the following fact.

Fact 3.3 (Modeling property for s-indiscernibles [8, Thm. 4.3])**.** *Let X be $\mathcal{T}_\alpha$ or $\mathcal{F}_{\alpha+1}$. For any $(b_f)_{f\in X}$ and any set A of **hyper**imaginaries, there is a family of tuples $(c_f)_{f\in X}$ that is s-indiscernible over A and* locally based on $b_{\in X}$ *(i.e., for any finite tuple $f_0\dots f_{n-1}$ from X and any neighborhood U of* $\operatorname{tp}(c_{f_0}\dots c_{f_{n-1}}/A)$ *(in the appropriate type space), there is a tuple $g_0\dots g_{n-1}$ from X such that $f_0\dots f_{n-1}\equiv^{\mathrm{qf}} g_0\dots g_{n-1}$ and $\operatorname{tp}(b_{g_0}\dots b_{g_{n-1}}/A)\in U$).*

Note that while Fact 3.3 is normally formulated for discrete logic, the corresponding statement in continuous logic follows easily from a very soft general argument: Given a metric structure M and a tree $(b_f)_{f\in X}$ of elements of M, find α large enough that M, $\operatorname{Th}(M)$, and $b_{\in X}$ are elements of V_α and apply [8, Thm. 4.3] to V_α as a discrete structure and get some A-s-indiscernible family $(c^*_f)_{f\in X}$ of elements of an elementary extension $V^*_\alpha\succeq V_\alpha$. These elements live inside a structure $M^*\in V^*_\alpha$ that is internally a model of $\operatorname{Th}(M)$. By taking the standard parts of each real-valued predicate in M^* and then completing with regards to the metric, we get a metric structure N that is an elementary extension of M. For each $f\in X$, let c_f be the image in N of c^*_f under the canonical map from M^* to N. It is straightforward to check that $(c_f)_{f\in X}$ is the required A-s-indiscernible family.

Before proving full existence for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$, we will need a lemma.

Lemma 3.4. *Fix α and $\gamma>\alpha$. Let $(e_f)_{f\in\mathcal{F}_{\gamma+1}}$ be an s-indiscernible family of real tuples over a set A of **hyper**imaginary parameters. Let $\lambda=|Ae_{\trianglerighteq\zeta^\gamma_\alpha}|+|T|$. Suppose that there is an ultraimaginary c_F such that $c_F\in\operatorname{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq\zeta^{\gamma+1}_\alpha})\cap\operatorname{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq 1\frown\zeta^\gamma_\alpha})$. Then there is a model M with $Ac_F\subseteq\operatorname{dcl}^{\mathrm{u}}(M)$ and $|M|\leq\lambda$ such that $(e_f)_{f\in\mathcal{F}_{\gamma+1}}$ is s-indiscernible over M.*

Proof. By Fact 3.3, we can find a set of real parameters B such that $|B|\leq|A|+|T|$, $A\subseteq\operatorname{bdd}^{\mathrm{heq}}(B)$, and $(e_f)_{f\in\mathcal{F}_{\gamma+1}}$ is s-indiscernible over B.

Let T' be a Skolemization of T with $|T'|=|T|$. Let $\mathbb{M}'$ be the monster model of T', which we may think of as an expansion of $\mathbb{M}$. By Fact 3.3, we can find $(b'_f)_{f\in\mathcal{F}_{\gamma+1}}$ locally based on $(e_f)_{f\in\mathcal{F}_{\gamma+1}}$ which is s-indiscernible over B (in T'). By considering an automorphism of $\mathbb{M}$ (in T), we may assume that $(b'_f)_{f\in\mathcal{F}_{\gamma+1}}$ actually is $(e_f)_{f\in\mathcal{F}_{\gamma+1}}$, so that $(e_f)_{f\in\mathcal{F}_{\gamma+1}}$ is s-indiscernible over B (in T').

Find an automorphism $\sigma\in\operatorname{Aut}(\mathbb{M}'/B)$ satisfying that for every $i<\omega$, $\sigma\cdot e_{\trianglerighteq\langle i+1\rangle_{\gamma+1}}=e_{\trianglerighteq\langle i\rangle_{\gamma+1}}$. Let M be the Skolem hull of $B\cup\sigma\cdot e_{\trianglerighteq\zeta^{\gamma+1}_\alpha}$. Note that $(e_f)_{f\in\mathcal{F}_{\gamma+1}}$ is s-indiscernible over M (and therefore the same is true in T). Furthermore, note that $|M|\leq\lambda$.

Let M_i be the Skolem hull of $Be_{\trianglerighteq i\frown\zeta^\gamma_\alpha}$ for both $i\in\{0,1\}$. Note that $c_F\in\operatorname{bdd}^{\mathrm{u}}_\lambda(M_1)$ and $|M_1|\leq\lambda$. Pass back to the theory T. Note that M, M_0, and M_1 are still models of T. By Corollary 1.8, there is an invariant equivalence relation G (in T) such that c_F and $[M_1]_G$ are interdefinable. Therefore we have that $[M_1]_G\in\operatorname{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq\zeta^{\gamma+1}_\alpha})\subseteq\operatorname{bdd}^{\mathrm{u}}(M_0)=\operatorname{dcl}^{\mathrm{u}}(M_0)$. Find an automorphism $\tau\in\operatorname{Aut}(\mathbb{M}/M_1)$

such that $\tau(M_0) = M$ (which exists by indiscernibility). τ witnesses that $[M_1]_G \in \operatorname{dcl}^{\mathrm{u}}(M)$ and therefore $c_F \in \operatorname{dcl}^{\mathrm{u}}(M)$, so M is the required model. $\square$

Now we are ready to prove full existence for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$, but we will take the opportunity to prove a certain technical strengthening which we will need later in the construction of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley trees.

Lemma 3.5. *If $(b_f)_{f \in \mathcal{T}_\alpha}$ is a tree of real elements that is s-indiscernible over a set of **hyper**imaginaries A, then there is a $\gamma > \alpha$ and a tree $(e_f)_{f \in \mathcal{T}_{\gamma+1}}$ such that*

- *$e_{\in \mathcal{T}_{\gamma+1}}$ is s-indiscernible over A,*
- *for each $f \in \mathcal{T}_\alpha$, $b_f = e_{\iota_{\alpha,\gamma+1}(f)}$, and*
- *$e_{\trianglerighteq \zeta_\alpha^{\gamma+1}} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A e_{\trianglerighteq 1 \frown \zeta_\alpha^\gamma}$.*

(Note that $e_{\trianglerighteq \zeta_\alpha^{\gamma+1}}$ is the original tree.)

Proof. If $b_{\in \mathcal{T}_\alpha} \in \operatorname{acl}(A)$, then the statement is trivial, so assume that $b_{\in \mathcal{T}_\alpha} \notin \operatorname{acl}(A)$.

Fix $\lambda = |Ab_{\in \mathcal{T}_\alpha}| + |T|$. By Proposition 2.3, we have that that $b_{\in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$ if and only if $\operatorname{bdd}^{\mathrm{u}}_\lambda(Ab_{\in \mathcal{T}_\alpha}) \cap \operatorname{bdd}^{\mathrm{u}}_\lambda(Ac) = \operatorname{bdd}^{\mathrm{u}}_\lambda(A)$ for any c. Let $\mu = |\operatorname{bdd}^{\mathrm{u}}_\lambda(Ab_{\in \mathcal{T}_\alpha}) \setminus \operatorname{bdd}^{\mathrm{u}}_\lambda(A)|^+$.

We will build a family $(e_f : f \in \iota_{\gamma+1,\mu}(\mathcal{T}_{\gamma+1}))$ inductively, where γ is some successor ordinal less than μ. By an abuse of notation, we will systematically conflate the sets $\iota_{\alpha,\mu}(\mathcal{T}_\alpha)$ and $\mathcal{T}_\alpha$ (and likewise for $\iota_{\alpha,\mu}(\mathcal{F}_{\alpha+1})$ and $\mathcal{F}_{\alpha+1}$) for all $\alpha < \mu$. Note that in general this will mean that $e_{\trianglerighteq \zeta_\beta^\mu}$ is the same thing as $e_{\in \mathcal{T}_\beta}$.

Let $e_f = b_f$ for all $f \in \mathcal{T}_\alpha$. Since $b_{\in \mathcal{T}_\alpha} \notin \operatorname{acl}(A)$, we can find a family $(d_f)_{f \in \mathcal{F}_{\alpha+1}}$ extending $e_{\in \mathcal{T}_\alpha}$ such that $(d_{\trianglerighteq \zeta_{\alpha+1}^\mu \frown i})_{i<\omega}$ is a non-constant A-indiscernible sequence. By Fact 3.3, we can define e_f for all $f \in \mathcal{F}_{\alpha+1}$ in such a way that the family $e_{\in \mathcal{F}_{\alpha+1}}$ is locally based on $d_{\in \mathcal{F}_{\alpha+1}}$. In particular, $(e_{\trianglerighteq \zeta_{\alpha+1}^\mu \frown i})_{i<\omega}$ will be a non-constant A-indiscernible sequence.

At successor stage $\beta + 1 \geq \alpha$, assume that we have defined e_f for all $f \in \mathcal{F}_{\beta+1}$ and that the family $(e_f)_{f \in \mathcal{F}_{\beta+1}}$ is s-indiscernible over A. If there is no $d_E \in \operatorname{bdd}^{\mathrm{u}}_\lambda(Ab_{\in \mathcal{T}_\alpha}) \setminus \operatorname{bdd}^{\mathrm{u}}_\lambda(A)$ such that the family $(e_f)_{f \in \mathcal{F}_{\beta+1}}$ is s-indiscernible over Ad, let $e_{\zeta_{\beta+1}^\mu} = \varnothing$ and $\gamma = \beta$ and halt the construction. Otherwise, let $e_{\zeta_{\beta+1}^\mu} = d$. For later reference, let $E_{\beta+1}$ be E. Note that the family $e_{\in \mathcal{T}_{\beta+1}}$ is s-indiscernible over A. Since $d_E \notin \operatorname{bdd}^{\mathrm{u}}_\lambda(A)$, we can find, by Proposition 1.4, a sequence $(\sigma_i)_{i<\omega}$ of elements of $\operatorname{Aut}(\mathbb{M}/A)$ such that $(\sigma_i \cdot d)_{i<\omega}$ is an A-indiscernible sequence satisfying $(\sigma_i \cdot d) \not E_{\beta+1} (\sigma_j \cdot d)$ for any $i < j < \omega$. Now choose $(e_f)_{f \in \mathcal{F}_{\beta+2}}$ in such a way that $e_{\in \mathcal{F}_{\beta+2}}$ extends what was already defined, is s-indiscernible over A, and is locally based on the family $(c_f)_{f \in \mathcal{F}_{\beta+2}}$ defined by $c_{i \frown f} = \sigma_i \cdot e_f$ for all $f \in \mathcal{T}_{\beta+1}$ (which is possible by Fact 3.3). In particular, note that for any $i < j < \omega$, we still have that $(e_{\zeta_{\beta+2}^\mu \frown i}, e_{\zeta_{\beta+2}^\mu \frown j}) \equiv_A (\sigma_0 \cdot d, \sigma_1 \cdot d)$ and so, in particular, $e_{\zeta_{\beta+2}^\mu \frown i} \not E_{\beta+1} e_{\zeta_{\beta+2}^\mu \frown j}$.

At limit stage β, we have constructed the family $(e_f)_{f \in \mathcal{T}_\beta}$. Note that this family is automatically s-indiscernible over A. Extend it to a family $e_{\in \mathcal{F}_{\beta+1}}$ that is s-indiscernible over A. (This is always possible by Fact 3.3.)

Claim. For any $\beta < \delta < \mu$, if $E_{\beta+1} = E_{\delta+1}$, then $e_{\zeta_{\beta+1}^\mu} \not E_{\beta+1} e_{\zeta_{\delta+1}^\mu}$.

Proof of claim. The sequence $(e_{\zeta_{\beta+2}^\mu \frown i})_{i<\omega}$ is $e_{\zeta_{\delta+1}^\mu}$-indiscernible. Since

$$e_{\zeta_{\beta+2}^\mu \frown 0} \not E_{\beta+1} e_{\zeta_{\beta+2}^\mu \frown 1},$$

it must be the case that $e_{\zeta_{\delta+1}^\mu} \not E_{\beta+1} e_{\zeta_{\beta+2}^\mu \frown i}$ for all $i < \omega$. $\square_{\text{claim}}$

Let g be the partial function taking β to $[e_{\zeta^\mu_{\beta+1}}]_{E_{\beta+1}}$. By the claim, this is an injection into $\mathrm{bdd}^{\mathrm{u}}_\lambda(Ab_{\in\mathcal{T}_\alpha}) \setminus \mathrm{bdd}^{\mathrm{u}}_\lambda(A)$. By the choice of μ, g's domain cannot be cofinal in μ, so the construction must have halted at some $\gamma < \mu$.

Extend $e_{\in\mathcal{T}_\gamma}$ to $e_{\in\mathcal{F}_{\gamma+1}}$ in such a way that the resulting family is s-indiscernible over A. Set $e_{\zeta^\mu_{\gamma+1}} = \varnothing$.

Claim. For any $c_F \in \mathrm{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq\zeta^\mu_\alpha}) \setminus \mathrm{bdd}^{\mathrm{u}}_\lambda(A)$, $c_F \notin \mathrm{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq 1\frown\zeta^\gamma_\alpha})$.

Proof of claim. Assume that there is some $c_F \in (\mathrm{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq\zeta^\mu_\alpha}) \cap \mathrm{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq 1\frown\zeta^\gamma_\alpha})) \setminus \mathrm{bdd}^{\mathrm{u}}_\lambda(A)$. By Lemma 3.4, we can find a model M with $Ac_F \subseteq \mathrm{dcl}^{\mathrm{u}}(M)$ and $|M| \leq \lambda$ such that $e_{\in\mathcal{F}_{\gamma+1}}$ is s-indiscernible over M. By Corollary 1.8, there is an invariant equivalence relation G such that c_F and $[M]_G$ are interdefinable. But this means that we could have chosen $[M]_G$ to be d_E at stage γ, contradicting the fact that the construction halted. Therefore no such c_F can exist. $\square_{\text{claim}}$

So, by the claim, we have that $\mathrm{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq\zeta^\mu_\alpha}) \cap \mathrm{bdd}^{\mathrm{u}}_\lambda(Ae_{\trianglerighteq 1\frown\zeta^\gamma_\alpha}) = \mathrm{bdd}^{\mathrm{u}}_\lambda(A)$. Therefore, by the choice of λ, $e_{\trianglerighteq\zeta^\mu_\alpha} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A e_{\trianglerighteq 1\frown\zeta^\gamma_\alpha}$, as required. $\square$

Theorem 3.6 (Full existence)**.** *For any set of* ***hyper****imaginaries A and real tuples b and c, there is $b' \equiv^{\mathrm{L}}_A b$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$.*

Proof. It is sufficient to show this in the special case that $b = c$. Specifically, given d and e, if we can find $d'e' \equiv_A de$ such that $d'e' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A de$, then we have $d' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A e$ by monotonicity. So fix a set of hyperimaginaries A and a real tuple b. Let B be a set containing realizations of all Lascar strong types extending $\mathrm{tp}(b/A)$. We can now apply Lemma 3.5 to the family $(B_f)_{f\in\mathcal{T}_0}$ with $B_\varnothing = B$ to get a family $(E_f)_{f\in\mathcal{T}_{\gamma+1}}$ such that $E_{\zeta_0^{\gamma+1}} = B$ for some $f \in \mathcal{T}_{\gamma+1}$, $B \equiv_A B_f$, and $B \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A B_f$. Let σ be an automorphism fixing A taking B_f to B. Let $B' = \sigma\cdot B$. B' still contains realizations of all Lascar strong types extending $\mathrm{tp}(b/A)$, so we can find $b' \in B'$ with $b' \equiv^{\mathrm{L}}_A b$, which is the required element. $\square$

Corollary 3.7. *For any set of* ***hyper****imaginaries A and any ultraimaginaries b_E and c_F, there is $b'_E \equiv^{\mathrm{L}}_A b_E$ such that $b'_E \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c_F$.*

Proof. Apply Theorem 3.6 to b and c to get $b' \equiv^{\mathrm{L}}_A b$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$. We then have that $\mathrm{bdd}^{\mathrm{u}}(b') \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A \mathrm{bdd}^{\mathrm{u}}(c)$, so by monotonicity, $b'_E \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c_F$. $\square$

Corollary 3.8 (Extension)**.** *For any set of* ***hyper****imaginaries A and any ultraimaginaries b_E, c_F, and d_G, if $b_E \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c_F$, then there is $b'_E \equiv^{\mathrm{L}}_{Ac_F} b_E$ such that $b'_E \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c_F d_G$.*

Proof. By Corollary 3.7, we can find $b'_E \equiv^{\mathrm{L}}_{Ac_F} b$ such that $b'_E \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_{Ac_F} d_G$. By symmetry and transitivity, we have that $b'_E \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c_F d_G$. $\square$

Compactness is very essential in the proof of Fact 3.3 and therefore also Theorem 3.6, which raises the following question.

Question 3.9. *Does Theorem 3.6 hold when A is a set of ultraimaginaries?*

4. Total $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{bu}}$-Morley sequences

Definition 4.1. A $\downfree^{\mathrm{bu}}$-*Morley sequence over A* is an A-indiscernible sequence $(b_i)_{i<\omega}$ such that $b_i \downfree^{\mathrm{bu}}_A b_{<i}$ for each $i < \omega$.

A *weakly total $\downfree^{\mathrm{bu}}$-Morley sequence over A* is an A-indiscernible sequence $(b_i)_{i<\omega}$ such that for any finite I and any J (of any order type), if $I + J \equiv^{\mathrm{EM}}_A b_{<\omega}$, then $I \downfree^{\mathrm{bu}}_A J$.[10]

A *total $\downfree^{\mathrm{bu}}$-Morley sequence over A* is an A-indiscernible sequence $(b_i)_{i<\omega}$ such that for any I and J (of any order type), if $I + J \equiv^{\mathrm{EM}}_A b_{<\omega}$, then $I \downfree^{\mathrm{bu}}_A J$.

We could write down stronger and weaker forms of the $\downfree^{\mathrm{bu}}$-Morley condition, but we are really only interested in total $\downfree^{\mathrm{bu}}$-Morley sequences, as they seem to be a fairly robust class (see Theorem 4.8). Weakly total $\downfree^{\mathrm{bu}}$-Morley sequences seem to be the best we can get without large cardinals, however, which does raise the following question.

Question 4.2. *Is every weakly total $\downfree^{\mathrm{bu}}$-Morley sequence a total $\downfree^{\mathrm{bu}}$-Morley sequence?*

One immediate property of total $\downfree^{\mathrm{bu}}$-Morley sequences is that they act as universal witnesses of the relation $\equiv^{\mathrm{L}}_A$ in a strong way.

Proposition 4.3. *For any A and b, if there is a total $\downfree^{\mathrm{bu}}$-Morley sequence $(b_i)_{i<\omega}$ over A with $b_0 = b$, then for any b', $b' \equiv^{\mathrm{L}}_A b$ if and only if there are $I_0, J_0, I_1, \dots, J_{n-1}, I_n$ such that $b \in I_0$, $b' \in I_n$, and, for each $i < n$, $I_i + J_i$ and $I_{i+1} + J_i$ are both A-indiscernible and have the same EM-type as $b_{<\omega}$.*

Proof. Let $I = (b_i)_{i<\omega}$. We only need to prove that if $b' \equiv^{\mathrm{L}}_A b$, then the required configuration exists (as the required configuration is clearly sufficient to witness that $b' \equiv^{\mathrm{L}}_A b$). Choose I' so that $bI \equiv^{\mathrm{L}}_A b'I'$. Extend I to an A-indiscernible sequence $I + J$ with $I \equiv_A J$. By assumption $I \downfree^{\mathrm{bu}}_A J$, so by Proposition 2.4, there are $I_0, J_0, I_1, J_1, \dots, J_{n-1}, I_n$ such that $I_0 = I$, $J_0 = J$, $I_n = I'$, and for each $i < n$, $I_i \equiv^{\mathrm{L}}_{AJ_i} I_{i+1}$ and $J_i \equiv^{\mathrm{L}}_{AI_{i+1}} J_{i+1}$ if $i < n$. Since $I_0 + J_0$ is A-indiscernible, we can show by induction that $I_i + J_i$ and $I_{i+1} + J_i$ are both A-indiscernible and have the same EM-type as $I_0 = b_{<\omega}$. □

A similar statement is true for weakly total $\downfree^{\mathrm{bu}}$-Morley sequences, which we will state in Corollary 4.18 after we have shown that weakly total $\downfree^{\mathrm{bu}}$-Morley sequences always exist without set-theoretic hypotheses.

4.1. Characterization of total $\downfree^{\mathrm{bu}}$-Morley sequences.

Definition 4.4. For any set of parameters A, we write $\approx_A$ for the transitive closure of the relation $I \sim_A J$ that holds if and only if I and J are both infinite A-indiscernible sequences (of real or hyperimaginary elements) and either $I + J$ or $J + I$ is an A-indiscernible sequence.

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[10]Note that if we modified this definition to allow I to be any order type and require that J be finite, the resulting sequences would be precisely the order-reversals of the weakly total $\downfree^{\mathrm{bu}}$-Morley sequences as we have defined the term here (by symmetry of $\downfree^{\mathrm{bu}}$).

By an abuse of notation, we write $[I]_{\approx_A}$ for the ultraimaginary $[AI]_E$, where E is the equivalence relation on tuples of the same length as AI such that $E(AI, BJ)$ holds if and only if $A = B$ in our fixed enumeration and $I \approx_A J$.

Note that we do not in general require that I and J have the same order type. Also note that $\approx_A$ is reflexive: For any infinite A-indiscernible sequence I, we can find an infinite sequence J such that $I + J$ is also A-indiscernible. Then $I \sim_A J \sim_A I$, so $I \approx_A I$.

We will also need an appropriate Lascar strong type generalization of Ehrenfeucht-Mostowski type.

Definition 4.5. Given two infinite A-indiscernible sequences I and J, we say that I and J have the same *Lascar-Ehrenfeucht-Mostowski type* (or *LEM-type*) *over* A, written $I \equiv^{\mathrm{LEM}}_A J$, if there is some $J' \equiv^{\mathrm{L}}_A J$ such that $I + J'$ is A-indiscernible.

To see that the name is justified, note that two infinite A-indiscernible sequences I and J have the same Ehrenfeucht-Mostowski type over A if and only if there is a $J' \equiv_A J$ such that $I + J'$ is A-indiscernible.

Lemma 4.6. *For any infinite order types O and O', $I \approx_A J$ if and only if there are $K_0, L_0, K_1, \dots, L_{n-1}, K_n$ such that*

- *$K_0 = I$ and $K_n = J$,*
- *for $0 < i < n$, K_i is a sequence of order type O,*
- *for $i < n$, L_i is a sequence of order type O', and*
- *for $i < n$, $K_i + L_i$ and $K_{i+1} + L_i$ are A-indiscernible.*

Proof. The $\Leftarrow$ direction is obvious.

For the $\Rightarrow$ direction, we will proceed by induction. First assume that $I \sim_A J$. If $I + J$ is A-indiscernible, then find L of order type O' such that $I + J + L$ is A-indiscernible. We then have that $I + L$ and $J + L$ are A-indiscernible. If $J + I$ is A-indiscernible, then find L of order type O' such that $J + I + L$ is A-indiscernible. We then have that $I + L$ and $J + L$ are A-indiscernible.

Now assume that we know the statement holds for any I and J such that there is a sequence $I'_0, \dots, I'_n$ with $I'_0 = I$, $I'_n = J$, and $I'_i \sim_A I'_{i+1}$ for each $i < n$. Now assume that there is a sequence $I'_0, \dots, I'_{n+1}$ with $I'_0 = I$, $I'_{n+1} = J$, and $I'_i \sim_A I'_{i+1}$ for each $i \leq n$. Apply the induction hypothesis to get $K_0, L_0, K_1, \dots, L_{m-1}, K_m$ satisfying the properties in the statement of the lemma with $K_0 = I$ and $K_m = I'_n$. Now since $I'_n \sim_A I'_{n+1} = J$, we can apply the $n = 1$ case to get L_m such that $I'_n + L_m$ and $I'_{n+1} + L_m$ are both A-indiscernible. By compactness, we can find K^*_m of order type O such that $K^*_m + L_m$ and $K^*_m + L_{m-1}$ are both A-indiscernible. We then have that $K_0, L_0, K_1, L_1, \dots, K_{m-1}, L_{m-1}, K^*_m, L_m, K_{m+1}$ is the require sequence, where $K_{m+1} = J$. □

Proposition 4.7. *Fix a set of hyperimaginary parameters A.*

(1) $\equiv^{\mathrm{LEM}}_A$ is an equivalence relation on the class of infinite A-indiscernible sequences.
(2) If I and J have the same order type, then $I \equiv^{\mathrm{L}}_A J$ if and only if $I \equiv^{\mathrm{LEM}}_A J$.
(3) If $I \equiv^{\mathrm{LEM}}_A J$, then $I \equiv^{\mathrm{EM}}_A J$.
(4) If $I \approx_A J$, then $I \equiv^{\mathrm{LEM}}_A J$.

Proof. Recall the following fact: If I and J have the same order type and $I+J$ is A-indiscernible, then $I \equiv^{\mathrm{L}}_A J$.[11]

(1). First, to see that $\equiv^{\mathrm{LEM}}_A$ is reflexive, note that if I is an infinite A-indiscernible sequence, then any infinite A-indiscernible extension $I+I'$ will witness that $I \equiv^{\mathrm{LEM}}_A I$. To see that $\equiv^{\mathrm{LEM}}_A$ is symmetric, assume that $I \equiv^{\mathrm{LEM}}_A J$, and let J' be as in the definition of $\equiv^{\mathrm{LEM}}$. Find I' such that $IJ' \equiv^{\mathrm{L}}_A I'J$. Then extend $I'+J$ to $I'+J+I''$, where I'' has the same order type as I. We then have that $I'' \equiv^{\mathrm{L}}_A I' \equiv^{\mathrm{L}}_A I$, so $J \equiv^{\mathrm{LEM}}_A I$. To see that $\equiv^{\mathrm{LEM}}_A$ is transitive, assume that $I \equiv^{\mathrm{LEM}}_A J$ and $J \equiv^{\mathrm{LEM}}_A K$. Let this be witnessed by J' and K' such that $I+J'$ and $J+K'$ are A-indiscernible. Find K'' with the same order type as K such that $I+J'+K''$ is A-indiscernible. Then find K^* such that $J'K'' \equiv^{\mathrm{L}}_A JK^*$. Note that both $J+K^*$ and $J+K'$ are A-indiscernible. By compactness, we can find K^{**} of the same order type as K such that $K^{**}+J+K^*$ and $K^{**}+J+K'$ are both A-indiscernible. By the above fact, we then have that $K^* \equiv^{\mathrm{L}}_A K^{**} \equiv^{\mathrm{L}}_A K'$. Finally, $K' \equiv^{\mathrm{L}}_A K$ by assumption, so we have that $K'' \equiv^{\mathrm{L}}_A K$ and therefore that $I \equiv^{\mathrm{LEM}}_A K$.

(2) is immediate from the fact. (3) is obvious.

For (4), it is sufficient to show that $I \sim_A J \Rightarrow I \equiv^{\mathrm{LEM}}_A J$. This follows immediately from the fact that $I \equiv^{\mathrm{L}}_A I$ and $J \equiv^{\mathrm{L}}_A J$. □

Now we will see that total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences over A are precisely those which are 'as generic as possible' in terms of $\approx_A$ (i.e., their $\equiv^{\mathrm{LEM}}_A$-equivalence class decomposes into a single $\approx_A$-equivalence class).

Theorem 4.8. *For any A-indiscernible sequence $(b_i)_{i<\omega}$ (with A a set of hyperimaginary parameters), the following are equivalent.*

1. *$b_{<\omega}$ is a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A.*
2. *There exists a pair of infinite sequences I and J (of any, possibly distinct order types) such that $I+J \equiv^{\mathrm{EM}}_A b_{<\omega}$ and $I \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A J$.*
3. *For any K, $K \approx_A b_{<\omega}$ if and only if $K \equiv^{\mathrm{LEM}}_A b_{<\omega}$.*
4. *$[b_{<\omega}]_{\approx_A} \in \mathrm{bdd}^{\mathrm{u}}(A)$.*

Proof. (1)⇒(2). This is immediate from the definition.

(2)⇒(3). First note that if $K \approx_A b_{<\omega}$, then $K \equiv^{\mathrm{LEM}}_A b_{<\omega}$ by Proposition 4.7. Let I and J be as in the statement of (2). By compactness, we may find $I' \equiv_A b_{<\omega}$ such that $I'+I+J$ is A-indiscernible. By applying an automorphism fixing A, we may assume that $b_{<\omega}+I+J$ is A-indiscernible. Fix K such that $K \equiv^{\mathrm{LEM}}_A b_{<\omega}$. By compactness, we can find a $K' \equiv_A K$ such that $b_{<\omega}+I+K'+J$ is A-indiscernible. We have that $K \equiv^{\mathrm{LEM}}_A b_{<\omega} \sim_A K'$ and therefore $K \equiv^{\mathrm{L}}_A K'$ by Proposition 4.7. Let $a_E \in \mathrm{bdd}^{\mathrm{u}}(AI)$ be an ultraimaginary satisfying $\mathrm{dcl}^{\mathrm{u}}(a_E) = \mathrm{bdd}^{\mathrm{u}}(AI)$. Likewise, let $b_F \in \mathrm{bdd}^{\mathrm{u}}(AJ)$ be an ultraimaginary satisfying $\mathrm{dcl}^{\mathrm{u}}(b_F) = \mathrm{bdd}^{\mathrm{u}}(AJ)$.[12] Since $\mathrm{dcl}^{\mathrm{u}}(a_F) \cap \mathrm{dcl}^{\mathrm{u}}(b_F) = \mathrm{bdd}(A)$, we have that $K \equiv_{\mathrm{dcl}^{\mathrm{u}}_\lambda(Ia_F) \cap \mathrm{dcl}^{\mathrm{u}}_\lambda(Jb_F)} K'$ for all λ.

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[11] To see this, assume that I and J have the same order type and $I+J$ is A-indiscernible for some set of hyperimaginary parameters. Let M be a model with $A \subseteq \mathrm{bdd}^{\mathrm{heq}}(M)$. We can find an M-indiscernible sequence $I'+J'$ finitely based on $I+J$. In particular, this will have $I'+J' \equiv_A I+J$. Therefore we can find a model $M' \equiv_A M$ such that $I+J$ is M'-indiscernible. We then have that $I \equiv_{M'} J$, whereby $I \equiv^{\mathrm{L}}_A J$.

[12] We can take a_E to be $[[\mathrm{Autf}(\mathbb{M}/AI)]]$ and b_F to be $[[\mathrm{Autf}(\mathbb{M}/AJ)]]$ by Definition 1.9 and Proposition 1.12.

Therefore, by Proposition 1.10, we can find a sequence $(I^i a^i J^i b^i K^i)_{i \leq n}$ satisfying that $I^0 a^0 = Ia$, $J^0 b^0 = Jb$, $K^0 = K'$, $K^n = K$, and for each $i < n$,

- if i is even, then $I^i a^i = I^{i+1} a^{i+1}$ and $J^i b^i K^i \equiv_A J^{i+1} b^{i+1} K^{i+1}$ and
- if i is odd, then $J^i b^i = J^{i+1} b^{i+1}$ and $I^i a^i K^i \equiv_A I^{i+1} a^{i+1} K^{i+1}$.

By induction, we have that $I^i + K^i + J^i$ is A-indiscernible for each $i \leq n$. We therefore have that

$$K' = K^0 \sim_A I^0 \sim_A J^1 \sim_A I^2 \sim_A J^3 \sim_A \cdots \sim_A L \sim_A K^n = K,$$

where L is either I^n or J^n. Therefore $K' \approx_A K$.

(3)⇒(1). Assume that for any $K \equiv^{\mathrm{LEM}}_A b_{<\omega}$, $K \approx_A b_{<\omega}$. Let I and J be infinite sequences satisfying $I + J \equiv^{\mathrm{EM}}_A b_{<\omega}$. By applying an automorphism fixing A to $I + J$, we may assume that $b_{<\omega} + I + J$ is A-indiscernible. Fix some $I' \equiv^{\mathrm{L}}_A I$. We have that $I' \equiv^{\mathrm{LEM}}_A b_{<\omega}$, which by assumption implies that $I' \approx_A b_{<\omega}$. Since $b_{<\omega} \sim_A I$, we have that $I \approx_A I'$. By Lemma 4.6, we can find $K_0, L_0, K_1, L_1, \ldots, L_{n-1}, K_n$ such that $K_0 = I$, $K_n = I'$, L_0 has the same order type as J, K_i has the same order type as I for each $i \leq n$, L_i has the same order type as J for each $i < n$, and $K_i + L_i$ and $K_{i+1} + L_i$ are A-indiscernible for each $i < n$. Let $K_{-1} = I$ and $L_{-1} = J$. We now have that for each non-negative $i < n$, $K_{i-1} \equiv^{\mathrm{L}}_{AL_{i-1}} K_i$ and $L_{i-1} \equiv^{\mathrm{L}}_{AK_i} L_i$.[13] Therefore $K_{-1}, L_{-1}, K_0, L_0, \ldots, L_{n-1}, K_n$ is precisely the kind of sequence needed to apply Proposition 2.4 (with the indices shifted down by 1). Since we can do this for any $I' \equiv^{\mathrm{L}}_A I$, we have that $I \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A J$.

(3)⇒(4). Let x be a tuple of variables in the same sorts as $b_{<\omega}$. There are at most $2^{|Ab_{<\omega}|+|T|}$ many Lascar strong types in x over A. (3) implies therefore that there are at most $2^{|Ab_{<\omega}|+|T|}$ many $\approx_A$ classes with representatives that realize $\mathrm{tp}(b_{<\omega}/A)$. Therefore $[c_{<\omega}]_{\approx_A} \in \mathrm{bdd}^{\mathrm{u}}(A)$ for any $c_{<\omega} \equiv_A b_{<\omega}$ and so a fortiori $[b_{<\omega}]_{\approx_A} \in \mathrm{bdd}^{\mathrm{u}}(A)$.

(4)⇒(3). Let $I \equiv^{\mathrm{LEM}}_A b_{<\omega}$. Find I' such that $I \equiv^{\mathrm{L}}_A I'$ and $b_{<\omega} + I'$ is A-indiscernible. Since $[b_{<\omega}]_{\approx_A} \in \mathrm{bdd}^{\mathrm{u}}(A)$, we must have, by Proposition 1.4, that there are at most $2^{|Ab_{<\omega}|+|T|}$ conjugates of $[b_{<\omega}]_{\approx_A}$ under $\mathrm{Aut}(\mathbb{M}/A)$. For any $I'' \equiv_A I'$, we can find $c_{<\omega} \equiv_A b_{<\omega}$ such that $I'' \sim_A c_{<\omega}$. Therefore there are at most $2^{|Ab_{<\omega}|+|T|}$ conjugates of $[I']_{\approx_A}$ under $\mathrm{Aut}(\mathbb{M}/A)$ as well, and so $[I']_{\approx_A} \in \mathrm{bdd}^{\mathrm{u}}(A)$ by Proposition 1.4 again. By Proposition 1.13, there must be an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}/A, [I']_{\approx_A})$ such that $\sigma \cdot I' = I$. Therefore $[I']_{\approx_A} = [I]_{\approx_A}$ and hence $I \approx_A b_{<\omega}$. □

4.2. **Building ((weakly) total) $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequences.** Given that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$ satisfies full existence, an immediate, familiar Erdős-Rado argument gives that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequences exist, but in the end we will need a technical strengthening of this result.

Proposition 4.9. *If $(b_f)_{f \in \mathcal{T}_\alpha}$ is a family of real elements that is s-indiscernible over a set of hyperimaginaries A, then there is a family $(c_f)_{f \in \mathcal{F}_{\alpha+1}}$ such that*

- *$c_{\in \mathcal{F}_{\alpha+1}}$ is s-indiscernible over A,*
- *$c_{\iota_{\alpha,\alpha+1}(f)} = b_f$ for each $f \in \mathcal{T}_\alpha$, and*
- *the sequence $(c_{\trianglerighteq \langle i \rangle})_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequence over A.*

[13]For $i = 0$, we have that $K_{-1} \equiv^{\mathrm{L}}_{AL_{-1}} K_0$ trivially, since $K_{-1} = I = K_0$.

Proof. Let κ be sufficiently large to apply Erdős-Rado to a sequence of tuples of the same length as $b_{\in \mathcal{T}_\alpha}$ over the set A.

Let $\gamma(0) = \alpha$. Let $c_f^0 = b_f$ for all $f \in \mathcal{T}_{\gamma(0)} = \mathcal{T}_\alpha$. Let $g_0 = \varnothing$ (as an element of $\mathcal{T}_\alpha$).

At successor stage $\beta + 1$, assume we have $(c_f^\beta)_{\mathcal{T}_{\gamma(\beta)}}$ which is s-indiscernible over A and which satisfies $c^\beta_{\iota_{\gamma(\delta),\gamma(\beta)}(f)} = c_f^\delta$ for all $\delta < \beta$. By Lemma 3.5, we can build a family $(c_f^{\beta+1})_{\mathcal{T}_{\gamma(\beta+1)}}$ (for some successor ordinal $\gamma(\beta+1) > \gamma(\beta)$) such that

- $(c_f^{\beta+1})_{f \in \mathcal{T}_{\gamma(\beta+1)}}$ is s-indiscernible over A,
- for each $f \in \mathcal{T}_{\gamma(\beta)}$, $c_f^\beta = c^{\beta+1}_{\iota_{\gamma(\beta),\gamma(\beta+1)}(f)}$, and
- $c^{\beta+1}_{\trianglerighteq \zeta^{\gamma(\beta+1)}_{\gamma(\beta)}} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A c^{\beta+1}_{\trianglerighteq 1 \frown \zeta^{\gamma(\beta+1)-1}_{\gamma(\beta)}}$.

Let $g_{\beta+1} \in \mathcal{T}^*_{\gamma(\beta+1)}$ be $1 \frown \zeta_\alpha^{\gamma(\beta+1)-1}$. Note that $g_{\beta+1} \trianglerighteq h$. Also note that by construction we have that

$$c^{\beta+1}_{\trianglerighteq g_{\beta+1}} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A \{c^{\beta+1}_{\trianglerighteq \iota_{\gamma(\delta),\gamma(\beta+1)}(g_\delta)} : \delta \in (\beta+1) \setminus \lim(\beta+1)\},$$

since $\iota_{\gamma(\delta),\gamma(\beta+1)}(g_\delta) \trianglerighteq \zeta^{\gamma(\beta+1)}_{\gamma(\beta)}$ for all non-limit $\delta < \beta + 1$.

At limit stage β, let $\gamma(\beta) = \sup_{\delta<\beta} \gamma(\delta)$ and let $(c_f^\beta)_{f \in \mathcal{T}_{\gamma(\beta)}}$ be the direct limit of $(c_f^\delta)_{f \in \mathcal{T}_{\gamma(\delta)}}$ for $\delta < \beta$. Leave g_β undefined.

Stop once we have $(c_f^\kappa)_{f \in \mathcal{T}_{\gamma(\kappa)}}$. Consider the sequence $(c^\kappa_{\trianglerighteq \iota_{\gamma(\beta),\gamma(\kappa)}(g_\beta)})_{\beta \in \kappa \setminus \lim \kappa}$.[14] By our choice of κ and a standard application of the Erdős-Rado theorem, we can find a family $(c_f)_{f \in \mathcal{F}_{\alpha+1}}$ such that the sequence $(c_{\trianglerighteq \langle i \rangle_\alpha})_{i<\omega}$ is A-indiscernible and for every increasing tuple $\bar{\imath} < \omega$, there is $\bar{\beta} \in \kappa \setminus \lim \kappa$ such that $c_{\trianglerighteq \langle i_0 \rangle_\alpha} \dots c_{\trianglerighteq \langle i_k \rangle_\alpha} \equiv_A c^\kappa_{\trianglerighteq \iota_{\gamma(\beta_0),\gamma(\kappa)}(g_{\beta_0})} \dots c^\kappa_{\trianglerighteq \iota_{\gamma(\beta_k),\gamma(\kappa)}(g_{\beta_k})}$.

In particular, note that this implies that

$$c_{\trianglerighteq \langle i \rangle_\alpha} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A \{c_{\trianglerighteq \langle j \rangle_\alpha} : j < i\}$$

for every $i < \omega$. Clearly by applying an automorphism, we may assume that $c_{\iota_{\alpha,\alpha+1}(f)} = b_f$ for each $f \in \mathcal{T}_\alpha$, so all we need to do is show that the family $c_{\in \mathcal{F}_{\alpha+1}}$ is s-indiscernible over A.

Since the sequence $(c_{\trianglerighteq \langle i \rangle_\alpha})_{i<\omega}$ is A-indiscernible, it is sufficient, by induction, to show the following statement: For any sequence $\bar{f}_0, \bar{f}_1, \dots, \bar{f}_k, \dots, \bar{f}_\ell$ of tuples of elements of $\mathcal{F}_{\alpha+1}$ satisfying $\bar{f}_i \trianglerighteq \langle i \rangle_\alpha$ for all $i \leq \ell$ and any $\bar{h} \trianglerighteq \langle k \rangle_\alpha$ such that $\bar{f}_k$ and $\bar{h}$ realize the same quantifier-free type, we have that $c_{\bar{f}_k}$ and $c_{\bar{h}}$ realize the same type over $Ac_{\bar{f}_0} \dots c_{\bar{f}_{k-1}} c_{\bar{f}_{k+1}} \dots c_{\bar{f}_\ell}$.

So let $\bar{f}_0, \dots, \bar{f}_\ell$ and $\bar{h}$ be as in the statement. By construction, there are $\beta_0, \dots, \beta_\ell$ such that $c_{\trianglerighteq \langle i \rangle_\alpha} \equiv_A c^\kappa_{\trianglerighteq \iota_{\gamma(\beta_i),\gamma(\kappa)}(g_{\beta_i})}$ for each $i \leq \ell$. Let $\bar{f}'_0, \dots, \bar{f}'_\ell, \bar{h}'$ be the corresponding elements of $\mathcal{T}_{\gamma(\kappa)}$. (So, in particular, $\bar{f}'_i \trianglerighteq g_{\beta_i}$ for each $i \leq \ell$ and $\bar{h}' \trianglerighteq g_{\beta_k}$). We now have that $\bar{f}'_k$ and $\bar{h}'$ realize the same quantifier-free type. Therefore, by the s-indiscernible of $c^\kappa_{\in \mathcal{T}_{\gamma(\kappa)}}$, we have that $c^\kappa_{\bar{f}'_k}$ and $c^\kappa_{\bar{h}'}$ realize the same type over $Ac^\kappa_{\bar{f}'_0} \dots c^\kappa_{\bar{f}'_{k-1}} c^\kappa_{\bar{f}'_{k+1}} \dots c^\kappa_{\bar{f}'_\ell}$. From this the required statement follows, and we have that $c_{\in \mathcal{F}_{\alpha+1}}$ is s-indiscernible over A. $\square$

[14] We write $\lim \alpha$ for the set of limit ordinals in α.

Corollary 4.10. *For any set of hyperimaginaries A and any real tuple b, there is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence $(b_i)_{i<\omega}$ over A with $b_0 = b$.*

Proof. Apply Proposition 4.9 to the tree $(b_f)_{f\in\mathcal{T}_0}$ defined by $b_\varnothing = b$.[15] □

The order type ω is essential, however; Erdős-Rado only guarantees the existence of sequences that satisfy the relevant condition on finite tuples. Fortunately, this is more than sufficient for the following weak 'chain condition.'

Lemma 4.11. *If $(b_i)_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A that is moreover Ac-indiscernible, then $b_0 \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$.*

Proof. Fix λ. Let $\mu = |\operatorname{bdd}^{\mathrm{u}}_\lambda(Ac) \setminus \operatorname{bdd}^{\mathrm{u}}_\lambda(A)|$. Extend $b_{<\omega}$ to $(b_i)_{i<\mu^+}$. We still have that for any $i < j < \mu^+$, $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A b_j$ (since this is only a property of $\operatorname{tp}(b_i b_j/A)$). Therefore the sets $\operatorname{bdd}^{\mathrm{u}}_\lambda(Ab_i) \setminus \operatorname{bdd}^{\mathrm{u}}_\lambda(A)$ are pairwise disjoint. Since there are μ^+ many of them, one of them must be disjoint from $\operatorname{bdd}^{\mathrm{u}}_\lambda(Ac) \setminus \operatorname{bdd}^{\mathrm{u}}_\lambda(A)$. Therefore by indiscernibility, we must have $b_0 \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$. □

We will not use the following corollary of Lemma 4.11, but it is worth pointing out.

Corollary 4.12. *If I is a total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A that is Ac-indiscernible, then $I \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$.*

Proof. Extend I to an Ac-indiscernible sequence $I_0 + I_1 + I_2 + \dots$ with $I_0 = I$. Since I is totally $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley, we have that $(I_i)_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A. So by Lemma 4.11, we have $I = I_0 \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$. □

Parts (2) and (3) of following definition are equivalent to [8, Def. 2.1, 3.4] in our context; this formulation is used implicitly in [6] and its equivalence to the standard definition is discussed in [6, Rem. 5.8]. The rest of it is based on [6, Def. 5.7], although we have had to modify the definition of restriction slightly in order to deal with limit ordinals more smoothly.

Definition 4.13. Fix a family $(b_f)_{f\in\mathcal{T}_\alpha}$.

(1) For $w \subseteq \alpha$, the *restriction of $\mathcal{T}_\alpha$ to the set of levels w* is given by
$$\mathcal{T}_\alpha{\restriction}w = \{f \in \mathcal{T}_\alpha : \min\operatorname{dom}(f) \in w,\ \beta \in \operatorname{dom}(f) \setminus w \Rightarrow f(\beta) = 0\}.$$
(2) A family $(b_f)_{f\in\mathcal{T}_\alpha}$ is *str-indiscernible over A* if it is s-indiscernible over A and satisfies that for any finite $w, v \subseteq \alpha \setminus \lim\alpha$ with $|w| = |v|$, $b_{\in\mathcal{T}_\alpha\restriction w}$ and $b_{\in\mathcal{T}_\alpha\restriction v}$ realize the same type over A (where we take $b_{\in\mathcal{T}_\alpha\restriction w}$ to be enumerated according to $<_{\mathrm{lex}}$, which is a well-ordering on $\mathcal{T}_\alpha{\restriction}w$ for finite w).
(3) We say that $b_{\in\mathcal{T}_\alpha}$ is $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$*-spread-out over A* if for any $f \in \mathcal{T}^*_\alpha$ (with $\operatorname{dom}(f) = [\beta+1, \alpha)$ for some $\beta < \alpha$), the sequence $(b_{\trianglerighteq f\frown i})_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A.
(4) We say that $b_{\in\mathcal{T}_\alpha\restriction w}$ is $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$*-spread-out over A* if for any $f \in \mathcal{T}^*_\alpha$ (with $\operatorname{dom}(f) = [\beta+1, \alpha)$ for some $\beta < \alpha$ and satisfying that $(f \frown i)_{i<\omega}$ is a sequence of elements of $\mathcal{T}_\alpha{\restriction}w$), the sequence $(b_{\trianglerighteq f\frown i})_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A (where we interpret b_f as $\varnothing$ if $f \notin \mathcal{T}_\alpha{\restriction}w$).

[15]This can also be proven directly by the standard argument for the existence of Morley sequences.

(5) $b_{\in\mathcal{T}_\alpha}$ is an $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-*Morley tree over* A if it is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out and *str*-indiscernible over A.

Note that if $b_{\in\mathcal{T}_\alpha}$ is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out over A, then any restriction $b_{\in\mathcal{T}_\alpha\restriction w}$ is also $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out over A (even for infinite w). Also note that, by a basic compactness argument, if α is infinite and $(b_f)_{f\in\mathcal{T}_\alpha}$ is *str*-indiscernible over A, then for any β, we can find a tree $(c_f)_{f\in\mathcal{T}_\beta}$ which is *str*-indiscernible over A such that for any $w\in[\alpha]^{<\omega}$ and $v\in[\beta]^{<\omega}$ with $|w|=|v|$, $b_{\in\mathcal{T}_\alpha\restriction w}\equiv_A c_{\in\mathcal{T}_\beta\restriction v}$.

Proposition 4.14. *For any set of hyperimaginaries A, real tuple b, and κ, there is a tree $(b_f)_{f\in\mathcal{T}_\kappa}$ that is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out and s-indiscernible over A such that for each $f\in\mathcal{T}_\kappa$, $b_f\equiv_A b$.*

Proof. Let $(b^0_f)_{f\in\mathcal{T}_0}$ be defined by $b^0_\varnothing=b$. This is vacuously $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out and s-indiscernible over A.

At successor stage $\alpha+1$, given $(b^\alpha_f)_{f\in\mathcal{T}_\alpha}$ which is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out and s-indiscernible by Proposition 4.9, we can find an extension $(b^{\alpha+1}_f)_{f\in\mathcal{F}_{\alpha+1}}$ satisfying $b^{\alpha+1}_{\iota_{\alpha,\alpha+1}(f)}=b^\alpha_f$ for all $f\in\mathcal{T}_\alpha$ such that $b^{\alpha+1}_{\in\mathcal{F}_{\alpha+1}}$ is s-indiscernible over A and $(b^{\alpha+1}_{\trianglerighteq\langle i\rangle_\alpha})_{i<\omega}$ is an $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-Morley sequence over A. By Fact 3.3, we can find $b^{\alpha+1}_\varnothing\equiv_A b$ such that the tree $(b^{\alpha+1}_f)_{f\in\mathcal{T}_{\alpha+1}}$ is s-indiscernible over A. By construction, we now have that $(b^{\alpha+1}_f)_{f\in\mathcal{T}_{\alpha+1}}$ is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out over A.

At limit stage α, let $(b^\alpha_f)_{f\in\mathcal{T}_\alpha}$ be the direct limit of $(b^\beta_f)_{f\in\mathcal{T}_\beta}$ for $\beta<\alpha$. It is immediate from the definitions that $b^\alpha_{\in\mathcal{T}_\alpha}$ is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out and s-indiscernible over A.

Once we have constructed $(b^\kappa_f)_{f\in\mathcal{T}_\kappa}$, let $b_f=b^\kappa_f$ for each $f\in\mathcal{T}_\kappa$. We have that $b_{\in\mathcal{T}_\kappa}$ is the required tree by induction. ☐

By the same argument as in [6, Lem. 5.10], we get the following.

Fact 4.15. *Fix a set of real parameters A, and let $(b_f)_{f\in\mathcal{T}_\kappa}$ be a family of tuples of real parameters of the same length that is s-indiscernible over A. If $\kappa\geq\beth_{\lambda^+}(\lambda)$ (where $\lambda=2^{|Ab_f|+|T|}$), then there is an str-indiscernible tree $(c_f)_{f\in\mathcal{T}_\omega}$ such that for any $w\in[\omega]^{<\omega}$, there is $v\in[\kappa]^{<\omega}$ such that*

$(*)_A$ *for any $w\in[\omega]^{<\omega}$, there is $v\in[\kappa]^{<\omega}$ such that $(b_f)_{f\in\mathcal{T}_\kappa\restriction v}\equiv_A(c_f)_{f\in\mathcal{T}_\omega\restriction w}$.*

Note that Fact 4.15 generalizes to continuous logic by the same soft argument as in the discussion after Fact 3.3.

Lemma 4.16. *Suppose that a family of tuples of real elements $(b_f)_{f\in\mathcal{T}_\kappa}$ is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out and s-indiscernible over a set of hyperimaginaries A with all b_f tuples of the same length. If $\kappa\geq\beth_{\lambda^+}(\lambda)$ (where $\lambda=2^{|Ab_f|+|T|}$), then there is an $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-Morley tree $(c_f)_{f\in\mathcal{T}_\omega}$ over A such that condition $(*)_A$ from Fact 4.15 holds.*

Proof. Find a model M with $|M|\leq|A|+\aleph_0$ such that $A\subseteq\mathrm{bdd}^{\mathrm{heq}}(M)$. Apply Fact 4.15 with M as the base to the family $(b_f)_{f\in\mathcal{T}_\kappa}$ to get a tree $(c_f)_{f\in\mathcal{T}_\omega}$ that is *str*-indiscernible over M and satisfies $(*)_M$. This is enough to imply that $c_{\in\mathcal{T}_\omega}$ is *str*-indiscernible over A and satisfies $(*)_A$. Furthermore, since the tree $c_{\in\mathcal{T}_\omega}$ has height ω and since $b_{\in\mathcal{T}_\kappa}$ is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out over A, $(*)_A$ implies that $c_{\in\mathcal{T}_\omega}$ is $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-spread-out over A. Therefore $c_{\in\mathcal{T}_\omega}$ is an $\mathop{\smile\hspace{-0.9em}|}^{\mathrm{bu}}$-Morley tree over A. ☐

Proposition 4.17. *If $(b_f)_{f\in\mathcal{T}_\omega}$ is a family of tuples of real elements that is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley tree over a set of hyperimaginaries A, then $(b_{\zeta^\omega_\beta})_{\beta<\omega}$ is a weakly total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A.*

Proof. Fix a linear order O. Let $c_\alpha = b_{\zeta^\omega_\alpha}$ for each $\alpha<\omega$.

For each positive $n<\omega$ and each $i<j<\omega$, we have that $b_{\trianglerighteq\zeta^\omega_n\frown i} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A b_{\trianglerighteq\zeta^\omega_n\frown j}$ and that the sequence $(b_{\trianglerighteq\zeta^\omega_n\frown i})_{i<\omega}$ is $Ac_{\geq n}$-indiscernible. By compactness, we can find $(c_i)_{i\in O}$ such that $(c_i)_{i\in\omega+O}$ is A-indiscernible and such that $(b_{\trianglerighteq\zeta^\omega_n\frown i})_{i<\omega}$ is $Ac_{\in[n,\omega)+O}$-indiscernible for each $n<\omega$.

Therefore, by Lemma 4.11, we have that $c_{<n}\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}_A c_{\in[n,\omega)+O}$. Hence, $(b_{\zeta^\omega_\beta})_{\beta<\omega}$ is a weakly total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence. □

Corollary 4.18. *For any set of hyperimaginaries A and tuple of real elements b, there is an A-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0=b$ such that for any $b'\equiv^{\mathrm{L}}_A b$ and $n<\omega$, there are $I_0,J_0,I_1,J_1\dots,J_{k-1},I_k$ with*

- *b the first element of I_0,*
- *b' the first element of I_k,*
- *$|I_i|=n$ for all $i\leq k$,*
- *J_i infinite for all $i<k$, and*
- *I_i+J_i and $I_{i+1}+J_i$ realizing the same EM-type over A as $b_{<\omega}$ for all $i<k$.*

We can also arrange it so that I_i is infinite for all $i\leq k$, $|J_i|=n$ for all $i<k$, and I_i+J_i and $I_{i+1}+J_i$ realize the same EM-type over A as $b_{<\omega}$ in the reverse order for all $i<k$ (with the same choice of $b_{<\omega}$ but possibly a different k).

Proof. By Lemma 4.16 and Proposition 4.17, we can find a sequence $(b_i)_{i<\omega}$ with $b_0=b$ that is a weakly total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A. Fix $n<\omega$, and write $b_{<\omega}$ as $I+J$ with $|I|=n$. By repeating the proof of Proposition 4.3, we get the required configuration of I_i's and J_i's.

For the final statement, by compactness, we can find an indiscernible sequence K of order type ω which has b as its first element and realizes the reverse of the EM-type of $b_{<\omega}$ over A. Fix an $n<\omega$. If we partition K as $I+J$ where $|J|=n$ and again repeat the proof of Proposition 4.3, we get the second required configuration of I_i's and J_i's. □

To go further, we will need the following fact from [10]. Recall that the statement $\kappa\to(\alpha)^{<\omega}_\gamma$ means that whenever $f:[\kappa]^{<\omega}\to\gamma$ is a function, there is a set $X\subseteq\kappa$ of order type α such that for each $n<\omega$, f is constant on $[X]^n$.

Fact 4.19 (Silver [10, Ch. 4])**.** *For any limit ordinal α, if κ is the smallest cardinal satisfying $\kappa\to(\alpha)^{<\omega}_2$, then for any $\gamma<\kappa$, $\kappa\to(\alpha)^{<\omega}_\gamma$. Furthermore, κ is strongly inaccessible.*

The smallest cardinal λ satisfying $\lambda\to(\alpha)^{<\omega}_2$ is called the Erdős cardinal $\kappa(\alpha)$. In the specific case of $\alpha=\omega$, we will also need the following lemma.

Lemma 4.20. *If $\kappa\to(\omega)^{<\omega}_\gamma$, then $(\gamma^\kappa)^+\to(\omega+1)^{<\omega}_\gamma$. In particular, if $\kappa(\omega)$ exists, then $(2^{\kappa(\omega)})^+\to(\omega+1)^{<\omega}_\gamma$ for any $\gamma<\kappa(\omega)$.*

Proof. Fix a set X of cardinality $(\gamma^\kappa)^+$ and a coloring $f:[X]^{<\omega}\to\gamma$. Fix an ordering $(x_\alpha)_{\alpha<(\gamma^\kappa)^+}$ of X. Recall that a subset $Y\subseteq X$ is *end-homogeneous* if for any $\delta_0<\dots<\delta_{n-1}<\alpha<\beta<(\gamma^\kappa)^+$, $f(\{x_{\delta_0},\dots,x_{\delta_{n-1}},x_\alpha\})=f(\{x_{\delta_0},\dots,x_{\delta_{n-1}},x_\beta\})$.

By [5, Lem. 15.2], there is an end-homogeneous set $Y \subseteq X$ of order type $\kappa+1$. Let $(y_\alpha)_{\alpha<\kappa+1}$ be an enumeration of Y in order. Let $g(A) = f(A \cup \{y_\kappa\})$. By assumption, there is a g-homogeneous subset $Z \subseteq Y$ of order type ω. Therefore, by construction, $Z \cup \{y_\kappa\}$ is the required f-homogeneous subset of order type $\omega+1$.

The last statement follows from the fact that $\kappa(\omega)$ is strongly inaccessible and cardinal arithmetic (i.e., $2^{\kappa(\omega)} = \gamma^{\kappa(\omega)}$ for $\gamma > 1$ with $\gamma < \kappa(\omega)$). □

Lemma 4.21. *Suppose $(b_f)_{f\in\mathcal{T}_\lambda}$ is $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-spread-out and s-indiscernible over A with all b_f tuples of the same length. If $\lambda \to (\omega+1)^{<\omega}_{2^{|Ab|+|T|}}$, then there is a set $X \subseteq \lambda \setminus \lim \lambda$ with order type $\omega+1$ such that $b_{\in\mathcal{T}_\lambda\restriction X}$ is an $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley tree over A.*

Proof. Let t be the function on $[\lambda \setminus \lim\lambda]^{<\omega}$ that takes $w \in [\lambda\setminus\lim\lambda]^{<\omega}$ to $\mathrm{tp}(b_{\in\mathcal{T}_\lambda\restriction w}/A)$. By assumption, we can find $X \subset \lambda\setminus\lim\lambda$ of order type $\omega+1$ such that t is homogeneous on X. $b_{\in\mathcal{T}_\lambda\restriction X}$ is s-indiscernible over A and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-spread-out over A, since these properties are both preserved by passing to restrictions. □

Theorem 4.22. *For any A and b in any theory T, if there is a cardinal λ satisfying $\lambda \to (\omega+1)^{<\omega}_{2^{|Ab|+|T|}}$, then there is a total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequence $(b_i)_{i<\omega}$ over A with $b_0 = b$.*

In particular, it is enough if there is an Erdős cardinal $\kappa(\alpha)$ such that $|Ab|+|T| < \kappa(\alpha)$ (for any limit $\alpha \geq \omega$).

Proof. If the Erdős cardinal $\kappa(\alpha)$ exists and $|Ab|+|T| < \kappa(\alpha)$, then by Fact 4.19, we have $2^{|Ab|+|T|} < \kappa(\alpha)$ as well. Then if $\alpha = \omega$, we have that $(2^{\kappa(\omega)})^+ \to (\omega+1)^{<\omega}_{2^{|Ab|+|T|}}$ by Lemma 4.20. If $\alpha > \omega$, we clearly have $\kappa(\alpha) \to (\omega+1)^{<\omega}_{2^{|Ab|+|T|}}$ by Fact 4.19. So in any such case we have the required λ.

Let λ be a cardinal such that $\lambda \to (\omega+1)^{<\omega}_{2^{|Ab|+|T|}}$ holds. By Proposition 4.14, we can build a tree $(b_f)_{f\in\mathcal{T}_\lambda}$ that is s-indiscernible and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-spread-out over A. By Lemma 4.21 and the choice of λ, we can extract an $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley tree $(c_f)_{f\in\mathcal{T}_{\omega+1}}$ from this.

By compactness, we can extend this to a tree $(c_f)_{f\in\mathcal{T}_{\omega+\omega}}$ that is str-indiscernible over A. We still have that for any $i<j<\omega$,

$$c_{\trianglerighteq\zeta^{\omega+\omega}_{\omega+1}\frown i} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A c_{\trianglerighteq\zeta^{\omega+\omega}_{\omega+1}\frown j}$$

but now we also have that the $(c_{\trianglerighteq\zeta^{\omega+\omega}_{\omega+1}\frown i})_{i<\omega}$ is $A\cup\{c_{\zeta^{\omega+\omega}_{\omega+i}} : i<\omega\}$-indiscernible, by str-indiscernibility of the full tree $c_{\in\mathcal{T}_{\omega+\omega}}$. Therefore, by Lemma 4.11,

$$c_{\trianglerighteq\zeta^{\omega+\omega}_{\omega+1}\frown 0} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A \{c_{\zeta^{\omega+\omega}_{\omega+i}} : i<\omega\},$$

so in particular,

$$\{c_{\zeta^{\omega+\omega}_{i}} : i<\omega\} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}_A \{c_{\zeta^{\omega+\omega}_{\omega+i}} : i<\omega\}.$$

Let $d_i = c_{\zeta^{\omega+\omega}_i}$ for each $i < \omega+\omega$. We have that $(d_i)_{i<\omega+\omega}$ is A-indiscernible. Furthermore, by Theorem 4.8, we have that $d_{<\omega}$ is a total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequence. By applying an automorphism, we get the required $b_{<\omega}$. □

So if we assume that for every λ, there is a κ such that $\kappa \to (\omega+1)^{<\omega}_\lambda$, we get that Lascar strong type is always witnessed by total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequences in the manner of Proposition 4.3.

The use of large cardinals in Theorem 4.22 leaves an obvious question.

Question 4.23. *Does the statement 'for every A and b, there is a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence $(b_i)_{i<\omega}$ over A with $b_0 = b$' have any set-theoretic strength? What if we add cardinality restrictions, such as $|A| + |T| \leq \aleph_0$ and $|b| < \aleph_0$?*

4.3. **Total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences in tame theories.** Lemma 4.11 can be used to show that $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}$ implies $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$ (where $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_A c$ means that $\mathrm{tp}(b/Ac)$ does not divide over A), something which was previously established for bounded hyperimaginary independence, $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{b}}$, in [4, Cor. 4.13] and which was originally folklore for algebraic independence, $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{a}}$.[16]

Proposition 4.24. *For any real elements A, b, and c, if $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_A c$, then $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$.*

Proof. Let $(c_i)_{i<\omega}$ be an $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A with $c_0 = c$. Since $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_A c$, we may assume that $c_{<\omega}$ is Ab-indiscernible. Therefore, by Lemma 4.11, $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A c$. □

Corollary 4.25. *If $(b_i)_{i<\omega}$ is a (non-dividing) Morley sequence over A, then it is a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A.* □

In simple theories, we get the converse (Proposition 4.27). Recall that $B \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{b}}_A C$ means $\mathrm{bdd}^{\mathrm{heq}}(AB) \cap \mathrm{bdd}^{\mathrm{heq}}(AC) = \mathrm{bdd}^{\mathrm{heq}}(A)$.

Lemma 4.26. *Let T be a simple theory. For any A, b, and c, $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A C$ if and only if there is an AC-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ such that for any J and K with $J + K \equiv^{\mathrm{EM}}_A b_{<\omega}$, $J \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{b}}_A K$.*

Proof. (The argument here is similar to the proof of [1, Lem. 3.2], but we will give a proof for the sake of completeness.) If $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A C$, then we can build an AC-indiscernible $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence $(b_i)_{i<\omega}$ over A with $b_0 = b$ (since T is simple). By some forking calculus, we have that $J \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_A K$ for any J and K with $J+K \equiv^{\mathrm{EM}}_A b_{<\omega}$. Therefore, by [4, Cor. 4.13], $J \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{b}}_A K$ for any such J and K as well.

Conversely, assume that there is an AC-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$ such that for any J and K with $J + K \equiv^{\mathrm{EM}}_A b_{<\omega}$, $J \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{b}}_A K$. Let κ be a regular cardinal such that every type (in the same sort as C) does not fork over some set of cardinality less than κ. Let $(b_i)_{i<\kappa+\kappa^*}$ be an AC-indiscernible sequence extending $b_{<\omega}$, where κ^* is an order-reversed copy of κ. Now we clearly have that $b_{<\kappa} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{b}}_A b_{\in\kappa^*}$. By local character, there is a set $D \subseteq Ab_{<\kappa}$ with $|D| < \kappa$ such that $C \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_D Ab_{<\kappa}$. Since κ is regular, there is a $\lambda < \kappa$ such that $D \subseteq Ab_{<\lambda}$. Therefore, by base monotonicity, $C \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{Ab_{<\lambda}} Ab_{<\kappa}$. Since $b_{\geq\lambda}$ is $Ab_{<\lambda}C$-indiscernible, we have that $C \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{Ab_{<\lambda}} Ab_{\in\kappa+\kappa^*}$. Therefore, by base monotonicity again, $C \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{Ab_{<\kappa}} Ab_{\in\kappa+\kappa^*}$. By the symmetric argument, $C \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{Ab_{\in\kappa^*}} Ab_{\in\kappa+\kappa^*}$ as well.

In simple theories, forking is characterized by canonical bases in the following way: $E \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_D F$ (with $D \subseteq F$) holds if and only if $\mathrm{cb}(\mathrm{tp}(E/\mathrm{bdd}^{\mathrm{heq}}(F))) \in \mathrm{bdd}^{\mathrm{heq}}(D)$ [7, Lem. 4.3.4]. Therefore, we have that $\mathrm{cb}(\mathrm{tp}(C/\mathrm{bdd}^{\mathrm{heq}}(Ab_{\in\kappa+\kappa^*}))) \in \mathrm{bdd}^{\mathrm{heq}}(Ab_{<\kappa}) \cap \mathrm{bdd}^{\mathrm{heq}}(Ab_{\in\kappa^*})$, but $\mathrm{bdd}^{\mathrm{heq}}(Ab_{<\kappa}) \cap \mathrm{bdd}^{\mathrm{heq}}(Ab_{\in\kappa^*}) = \mathrm{bdd}^{\mathrm{heq}}(A)$

[16] There is an incorrect proof of this in the literature. To the author's knowledge, the first correct published proof of this is in [4, Thm. 4.11].

by assumption. So $C \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{A} b_{\in \kappa+\kappa^*}$, whence $C \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{A} b_0$ and hence $b_0 \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{A} C$, as required. ☐

Proposition 4.27. *Let T be a simple theory. For any A and A-indiscernible sequence I, the following are equivalent.*

(1) I is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence over A.
(2) For any J and K with $J + K \equiv^{\mathrm{EM}}_{A} I$, $J \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{b}}_{A} K$.
(3) I is a total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A.

Proof. (1)⇒(3) is Corollary 4.25. (3)⇒(2) is obvious. For (2)⇒(1), assume that (2) holds. Fix $(b_i)_{i<\omega+\omega} \equiv^{\mathrm{EM}}_{A} I$. $(b_i)_{\omega \leq i < \omega+\omega}$ is $Ab_{<\omega}$-indiscernible. Therefore by Lemma 4.26, $b_\omega \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{A} b_{<\omega}$, and we have that $b_{<\omega+\omega}$, and therefore I, is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}$-Morley sequence over A. ☐

On the other hand, there are easy examples in NIP theories (such as DLO) of total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences that are not strict Morley sequences (i.e., sequences $b_{<\omega}$ satisfying that $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{A} b_{<i}$ and $b_{<i} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{A} b_i$ for all $i < \omega$). Fix a model M of DLO and let $(a_i b_i)_{i<\omega}$ be a sequence of elements above M satisfying $a_i < a_{i+1} < b_{i+1} < b_i$ for all $i < \omega$. This is a total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence since it is generated by an M-invariant type, but it is clearly not a strict Morley sequence. DLO can also be used to show that not every $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$-Morley sequence in a rosy theory is a total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence (e.g., [1, Ex. 3.13] is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$-Morley sequence since þ-forking in DLO is trivial but fails to even be an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{b}}$-Morley sequence).

In NSOP$_1$ theories, we do get that tree Morley sequences are total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences.

Proposition 4.28. *Let T be an NSOP$_1$ theory, and let $M \models T$. If I is a tree Morley sequence over M, then it is a total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over M.*

Proof. Let J be a sequence realizing the same EM-type as I over M. Find $K \equiv_M I$ such that $K \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{K}}_{M} IJ$. Let I', J', and K' have the same order type such that $I + I'$, $J + J'$, and $K + K'$ are all M-indiscernible. Since these are tree Morley sequences, we have that $I \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{K}}_{M} I'$, $J \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{K}}_{M} J'$, and $K \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{K}}_{M} K'$. Therefore, by the independence theorem for NSOP$_1$ theories, we can find I'' and J'' such that $I + I''$, $K + I''$, $K + J''$, and $J + J''$ are all M-indiscernible, so $I \approx_M J$.

Since we can do this for any such J, we have that I is a total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence by Theorem 4.8 and the fact that Lascar strong types are types over models. ☐

The converse is unclear. The argument in the context of simple theories relies on the existence of canonical bases for types.

Question 4.29. *If T is NSOP$_1$, is every total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over $M \models T$ a tree Morley sequence over M?*

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