Strongly minimal sets in continuous logic and Baldwin-Lachlan style characterizations of some inseparably categorical metric theories

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Background

Uncountably Categorical Theories

Definition

A theory is κ*-categorical* if it has a unique model of cardinality κ up to isomorphism.

- Morley proved that if a countable theory is categorical in some uncountable cardinality, then it is categorical in every uncountable cardinality. (So they're called *uncountably categorical*.)
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- Proof of Morley's theorem provides some but not a lot of structural information about uncountably categorical theories.
- Uncontably categorical theories are ω-stable and have no Vaughtian pairs.

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- Strongly minimal sets are categorical in every uncountable cardinality (have a good dimension theory like pure sets, vector spaces, and algebraically closed fields), therefore T is uncountably categorical.
- With some work you can show that if D and E are 'similar' strongly minimal sets over a countable model, then they have the same dimension. This implies that the theory has either 1 or ω countable models.

Continuous Logic

- Generalization of first-order logic to the context of *metric structures*: Underlying sets are complete metric spaces of bounded diameter with uniformly continuous $[0, 1]$-valued predicates.
- Quantifiers are sup and inf. Connectives are arbitrary continuous functions $F : [0, 1]^k \to [0, 1]$ for $k \leq \omega$.
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Definition

A *definable set* D is a set whose distance predicate $d(x,D)$ is a definable predicate.

Not every definable predicate's zeroset is a definable set.

Definition

A (type-definable) zeroset or type is *algebraic* if it is metrically compact in every model.

Inseparably Categorical Theories

Theorem (Ben Yaacov; Shelah, Usvyatsov)

Morley's theorem holds in continuous logic.

- A metric space with uncountable density character is called *inseparble*, so such theories are *inseparably categorical*.

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Inseparably categorical theories are ω-stable (need to count types with regards to metric density character).

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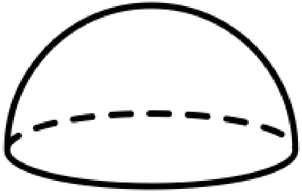

The type space $S_1(\mathfrak{H})$ for a typical infinite dimensional Hilbert space $\mathfrak{H}$. (Not drawn topologically.)

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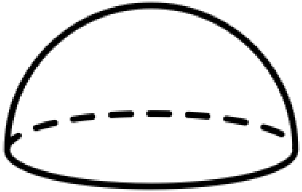

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- *IHS* does not have any strongly minimal types (see picture).
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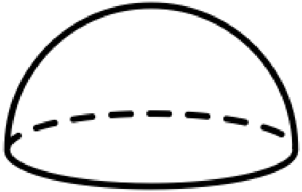
- What happened? *IHS* is ω-stable in the continuous sense, but in some ways it behaves like a discrete strictly stable theory (infinite forking chains).



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Baldwin-Lachlan in the presence of strongly minimal types

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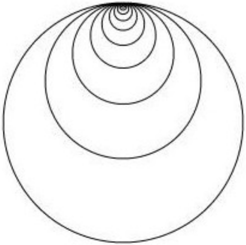

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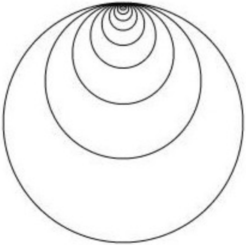
Proposition (H.)

$\mathrm{Th}(\mathbb{R}, +)$ *does not even interpret an infinite discrete theory.*



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- Yes.

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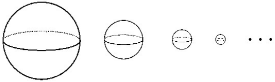

The structure $\mathfrak{A}$: Disjoint union of *IHS* spheres of radius 2^{-n} for $n < \omega$ where the distance between spheres is 1.

The type space $S_1(\varnothing)$ of $\mathrm{Th}(\mathfrak{A})$, topologically homeomorphic to $\omega + 1$. Limit point is strongly minimal but no $\varnothing$-definable set is.

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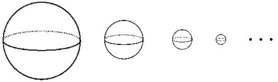
- Nevertheless, it looks 'approximately strongly minimal,' doesn't it?



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It does look approximately strongly minimal.

Definition

A zeroset F over the set A is $(<\varepsilon)$-*algebraic* if for every model $\mathfrak{M} \supseteq A$, $F(\mathfrak{M})$ can be covered by finitely many open ε-balls.

Definition

(D, P), with D a non-algebraic definable set and P a definable predicate, is an *approximately strongly minimal pair* if $\inf_{x \in D} P(x) = 0$ and for every pair $F, G \subseteq D$ of disjoint zerosets and every $\varepsilon > 0$, at least one of $F \cap [P \leq \varepsilon]$ and $G \cap [P \leq \varepsilon]$ is $(<\varepsilon)$-algebraic.

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Proposition (H.)

If (D, P) is an approximately strongly minimal pair, then $D \cap [P = 0]$ contains a unique strongly minimal type.

Finding approximately minimal pairs

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- Why are neither of these counterexamples where definable sets are behaving poorly occurring with ω-stable theories? Can we get a better handle on definable sets in ω-stable theories than in arbitrary theories?

Dictionaric Theories

Definition

A type space $S_n(A)$ is *dictionaric* if for every type p and open neighborhood U there exists a definable set D such that $p \in \mathrm{int} D \subseteq D \subseteq U$.
A theory T is *dictionaric* if all of its type spaces are.

Obviously every discrete theory is dictionaric.

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If $p \in S_n(A)$ is a strongly minimal type and $S_n(A)$ is dictionaric, then there is an A-definable strongly minimal pair (D, P) pointing to p.

Minimal Sets and Vaughtian Pairs

Definition

Given a model $\mathfrak{A}$, an $\mathfrak{A}$-definable non-algebraic set D is *minimal* if for every pair of disjoint $\mathfrak{A}$-zerosets $F, G \subseteq D$, at most one of $F(\mathfrak{A})$ or $G(\mathfrak{A})$ is non-compact.

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- Is it still true that in a theory with no Vaughtian pairs minimal sets are always strongly minimal?
- Maybe? What we do have is this:

Proposition (H.)

If T is a dictionaric theory with no Vaughtian pairs, then minimal sets are strongly minimal.

- But this is fine for our purposes because inseparably categorical theories are dictionaric.

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For every $n \leq \omega$ there is an inseparably categorical theory T_n with a $\varnothing$-definable strongly minimal imaginary I such that $\dim(I)$ can be anything $\leq \omega$ in the separable models of T_n but $S_1(\mathfrak{A})$ has a strongly minimal type if and only if $\dim(I(\mathfrak{A})) \geq n$.

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Proposition (H.)

If T is an inseparably categorical theory with a discrete strongly minimal imaginary then it has a discrete strongly minimal imaginary over the prime model.

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- If T is an inseparably categorical theory that has strongly minimal types over the prime model do we get Baldwin-Lachlan?
- Yes, partially:

Theorem (H.)

If T is a countable dictionaric theory such that T has a prime model with a minimal set (resp. imaginary) over it, then T is inseparably categorical if and only if it has no Vaughtian pairs (resp. no imaginary Vaughtian pairs). Furthermore such a theory has $\leq \omega$ many separable models.

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- Which, of course, raises the question:

When can we find strongly minimal types?

Two Axes of Difficulty

Continuous logic introduces two new difficulties:

- Lack of local compactness (of models).
- Lack of total disconnectedness (of type spaces).

IHS has both of these issues, but can we tackle one of them at a time?

Theories with a Locally Compact Model

This case is easy.

Proposition (H.)

If T has a locally compact model, then it is inseparably categorical if and only if it is ω-stable and has no Vaughtian pairs.
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Proof.

Only need to show $\Leftarrow$ direction, if T has a locally compact model and is ω-stable, then the prime model $\mathfrak{A}$ is locally compact. This means that $\mathfrak{A}$ is open as a subset of $S_1(\mathfrak{A})$, so $X = S_1(\mathfrak{A}) \setminus \mathfrak{A}$ is closed. Since T is ω-stable we can find a d-isolated-in-X type p. Since T is ω-stable it is dictionaric, so we can find a definable set D such that $D \cap X = \{p\}$. D has a unique non-algebraic type (over $\mathfrak{A}$), so it is minimal. Now use the theorem. □

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Definition

A metric space (X, d) is an ultrametric space if $d(x, z) \leq \max(d(x, y), d(y, z))$.

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Proposition (H.)

A theory T has totally disconnected type spaces if and only if it is dictionaric and has a $\varnothing$-definable ultrametric with scattered distance set that is uniformly equivalent to the metric.
Furthermore such theories are bi-interpretable with many-sorted discrete theories.

Not all ultrametric theories are dictionaric.

Baldwin-Lachlan for Ultrametric Theories

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An ω-stable theory has totally disconnected type spaces if and only if there is a $\varnothing$-definable ultrametric uniformly equivalent to the metric with distance set $\subseteq \{0\} \cup \{2^{-n}\}_{n<\omega}$.

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Proposition

If a theory T is ultrametric or has totally disconnected type spaces, then it is inseparably categorical if and only if it is ω-stable and has no imaginary Vaughtian pairs.
Furthermore such a theory has 1 or ω many separable models.

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- Idea of construction: Infinite wreath product of copies of a vector space over a finite field with 0 forgotten (i.e. underlying metric space is V^ω). Add predicates to relate vector spaces on even levels to each other and vector spaces on odd levels to each other, but leave the even and odd levels independent. Theory has 2 independent dimensions, so isn't inseparably categorical, but is ω-stable and with careful analysis of type space can be seen to have no Vaughtian pairs$^+$.

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- **Literal translation of the Baldwin-Lachlan theorem fails in continuous logic.**

The Number of Separable Models

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- What's going on with $\leq \omega$ rather than $\in \{1, \omega\}$?
- Some easy cases (T an inseparably categorical theory):
  - If T has a $\varnothing$-definable approximately minimal set/imaginary, then it has 1 or ω separable models.
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- There's also a relevant general result:

Theorem (Ben Yaacov, Usvyatsov)

If T is a countable superstable theory with 'enough uniformly d-finite types,' then it has 1 or $\geq \omega$ many separable models.

So if T has a strongly minimal set/imaginary over the prime model and has 'enough uniformly d-finite types,' then it has 1 or ω separable models.

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- The problem is precisely that $\mathrm{tp}(\alpha)$ fails to be d-finite, furthermore:

Proposition (H.)

There is a totally categorical theory T with a strongly minimal set over the unique separable model $\mathfrak{A}$, but such that for any strongly minimal $D(x, \bar{a})$ (in the home sort) and any $n \leq \omega$ there is $\bar{b} \equiv \bar{a}$ in $\mathfrak{A}$ such that $\dim(D(\mathfrak{A}, \bar{b})) = n$.

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- What if you have a strongly minimal set $D(x, \bar{a})$ such that $\mathrm{tp}(\bar{a})$ is d-finite?
- Unclear, proof is too delicate to generalize.
- What we do have right now is:

Theorem (H.)

If T is an inseparably categorical theory with a strongly minimal set $D(x, \bar{a})$ over the prime model such that $\mathrm{tp}(\bar{a})$ is d-finite, then T does not have precisely 2 separable models.

- Note that there are ω-stable continuous theories with precisely 2 separable models.

Thank you