An introduction to continuous logic

James Hanson

University of Maryland, College Park

April 15, 2022

VCU Analysis, Logic and Physics Seminar

Filters

Definition

A *filter* on a set I is a collection $\mathcal{F} \subseteq 2^I \setminus \{\varnothing\}$ that is closed under supersets and finite intersections.
A filter is *principal* if it contains a singleton.

- A filter represents a notion of 'largeness.' Example: The collection of cofinite subsets of an infinite set I.
- Filters were originally(?) introduced by Bourbaki as a generalization of sequences in the context of topology:

Definition

For a topological space (X, τ), a filter $\mathcal{F}$ on X *converges to the point* x if for every open neighborhood $U \ni x$, $U \in \mathcal{F}$.

(X, τ) is Hausdorff if and only if every filter converges to at most one point.

Ultrafilters

Definition

An *ultrafilter* on I is a filter $\mathcal{F}$ with the property that for any $J \subseteq I$, either $J \in \mathcal{F}$ or $I \setminus J \in \mathcal{F}$.

- Usually we mean non-principal ultrafilters.
- (X, τ) is compact if and only if every ultrafilter converges.
- Every filter can be extended to an ultrafilter (originally(?) shown by Tarski). Proof: Zorn's lemma.
- While originally motivated by topology and set theory, ultrafilters soon found applications in model theory (Skolem and later Łoś) but also various parts of analysis.

Discrete structures

Definition

A *language* $\mathcal{L}$ is a collection of constant symbols, relation symbols, and function symbols (with prescribed arities).
A *(discrete) $\mathcal{L}$-structure* is a set M together with interpreations of the constant, relation, and function symbols in $\mathcal{L}$.

- Example: Ordered fields can are $\mathcal{L}_{of}$-structures where $\mathcal{L}_{of}$ contains the constant symbols 0 and 1, the binary relation symbol $\leq$, the unary function symbol $-x$, and the binary function symbols $x + y$ and $x \cdot y$.
- $\mathbb{R}$ and $\mathbb{Q}$ are ordered fields.

Definition

Two $\mathcal{L}$-structures M and N are *isomorphic* if there is a function $f : M \to N$ that respects all constant, relation, and function symbols of $\mathcal{L}$.

$\mathbb{R}$ and $\mathbb{Q}$ are *not* isomorohpic $\mathcal{L}_{of}$-structures.

Ultraproducts and ultrapowers

- Fix a family $(M_i)_{i \in I}$ of $\mathcal{L}$-structures and an ultrafilter $\mathcal{F}$ on I.
- Consider the set $\prod_{i \in I} M_i$ and the equivalence relation
$$a \sim b \Leftrightarrow \{i \in I : a(i) = b(i)\} \in \mathcal{F}.$$
- $\sim$ respects relation and function symbols, so we can naturally regard $M_{\mathcal{F}} := \prod_{i \in I} M_i / \sim$ as an $\mathcal{L}$-structure. For example, $R(a)$ holds in $M_{\mathcal{F}}$ if and only if $\{i \in I : R(a_i) \text{ holds in } M_i\} \in \mathcal{F}$.
- $M_{\mathcal{F}}$ is the *ultraproduct of* $(M_i)_{i \in I}$ *(using* $\mathcal{F}$*)*.
- If $M_i = M$ for all $i \in I$, it is called an *ultrapower* and is written $M^{\mathcal{F}}$. There is a natural embedding ι of M into $M^{\mathcal{F}}$ given by taking $a \in M$ to the $\sim$-class of the constant function a.
- Intuition: Configurations that are finitely approximated in M are realized in $M^{\mathcal{F}}$. Example: If $N = (\mathbb{N}, \leq)$, then $N^{\mathcal{F}}$ will (usually) contain an element a satisfying $n \leq a$ for all $n \in \mathbb{N}$.

Doing discrete model theory backwards

- The Keisler-Shelah theorem allows us to characterize first-order logic in terms of ultrapowers.
- Fix a class $\mathbb{K}$ of $\mathcal{L}$-structures that is closed under ultraproducts.

Definition

Given $\mathcal{L}$-structures M, N and n-tuples $\bar{a} \in M$ and $\bar{b} \in N$, we say that $(M, \bar{a})$ and $(N, \bar{b})$ have the same *n-type*, and we write $(M, \bar{a}) \equiv (N, \bar{b})$, if there are ultrafilters $\mathcal{F}$ and $\mathcal{G}$ and an isomorphism $f : M^{\mathcal{F}} \to N^{\mathcal{G}}$ such that $f(\iota(\bar{a})) = \iota(\bar{b})$.

- For any $(M, \bar{a})$ and ultrafilter $\mathcal{F}$, we have that $(M, \bar{a}) \equiv (M^{\mathcal{F}}, \iota(\bar{a}))$.
- $\equiv$ is an equivalence relation. We write $S_n(\mathbb{K})$ to represent the collection of $\equiv$-classes of n-tuples, which we call n-types. We write $\mathrm{tp}(M, \bar{a})$ for the $\equiv$-class of $(M, \bar{a})$.

Type spaces in discrete logic

- $S_n(\mathbb{K})$ is a set.
- $\equiv$ is compatible with ultraproducts in the following sense: If $(M_i, \bar{a}_i) \equiv (N_i, \bar{b}_i)$ for each $i \in I$, then $(M_\mathcal{F}, \bar{a}_\mathcal{F}) \equiv (N_\mathcal{F}, \bar{b}_\mathcal{F})$ for any ultrafilter $\mathcal{F}$ on I.
- Induces a topology on $S_n(\mathbb{K})$ which is compact, Hausdorff, and totally disconnected (i.e., $S_n(\mathbb{K})$ is the Stone space of a Boolean algebra).
- An *n-formula* is a clopen subset of $S_n(\mathbb{K})$. Can also be thought of as a continuous $\{0, 1\}$-valued function.
- We write an n-formula as $\varphi(\bar{x})$ or φ.
- If $\mathrm{tp}(M, \bar{a})$ is in $\varphi(\bar{x})$, we write $M \models \varphi(\bar{a})$.

Operations on formulas

- We write $\varphi \wedge \psi$ for the intersection of φ and ψ, $\varphi \vee \psi$ for the union, and $\neg\varphi$ for the complement.
- There is a natural map $\pi : S_{n+1}(\mathbb{K}) \to S_n(\mathbb{K})$ which takes $\mathrm{tp}(M, \bar{a}b)$ to $\mathrm{tp}(M, \bar{a})$.
- This map is continuous and open, so for any $(n+1)$-formula $\varphi(\bar{x}, y)$, the image $\pi(\varphi(\bar{x}, y))$ is an n-formula. We write the projection as $\exists y \varphi(\bar{x}, y)$.
- The suggestions of this notation are correct (e.g., $M \models \varphi \wedge \psi(\bar{a})$ iff $M \models \varphi(\bar{a})$ and $M \models \psi(\bar{a})$, $M \models \exists y \varphi(\bar{a}, y)$ if and only if there is a $b \in M$ such that $M \models \varphi(\bar{a}, b)$, etc.).
- Basic expressions from the language involving the variables $\bar{x}$ are equivalent to formulas (e.g., when $\mathbb{K}$ is the class of ordered fields, there is a formula $\varphi(x_1 x_2 x_2)$ such that $M \models \varphi(abc)$ if and only if $a + 1 \leq b \cdot c$). These are called *atomic* formulas.
- The operations on this slide generate all n-formulas for every n.

Continuous logic

Ultraproducts in analysis

- At some point analysts realized that ultraproduct constructions had applications in analysis.
- The first(?) instance of this was in Gromov's proof that finitely generated groups of polynomial growth are nilpotent-by-finite.
- Some other applications: Asymptotic properties of Banach spaces. Many applications in C*-algebras and von Neumann algebras (Connes embedding problem). Construction of spherically complete non-Archimedean valued fields.
- Same intuition as before applies. Example: Dvoretzky's theorem says that in infinite-dimensional Banach spaces, there are arbitrarily good approximations of Hilbert spaces of arbitrarily high dimension. This can equivalently be equivalently stated (without explicit numerical bounds) as: For any infinite-dimensional Banach space X and (a typical) ultrafilter $\mathcal{F}$, $B^{\mathcal{F}}$ contains an infinite-dimensional subspace isometrically isomorphic to a Hilbert space.

Can we turn this into a kind of logic?

- As early as the 60s, model theorists realized that there should be a generalization of model theory to deal natively with real-valued structures. Early work by Chang and Keisler on very general classes of structures. Later work by Henson and Iovino (on Banach spaces in particular) and then by Ben Yaacov and Usvyatsov.
- Modern formulation was developed by Ben Yaacov, Berenstein, Henson, and Usvyatsov. (Although, frankly, Chang and Keisler deserve more credit.)
- I will present continuous logic in the same way I presented discrete logic, not because it is a good way to formalize continuous logic, but because it preserves more intuition from discrete model theory. (Also emphasizes the motivation from existing ultraproduct constructions in analysis.)
- Broad slogan: Whenever there is a notion of taking ultraproducts, there is a 'first-order theory' lurking around.

Metric structures

Definition (slightly non-standard)

A *(metric) language* $\mathcal{L}$ is a collection of constant symbols, relation symbols, and function symbols (with prescribed arities).
A *(metric) $\mathcal{L}$-structure* is a complete, bounded metric space (M, d) together with interpreations of the constant, relation, and function symbols in $\mathcal{L}$. Relations are boundedly $\mathbb{R}$-valued. Relations and functions are uniformly continuous.

- Requiring bounded metric spaces is a convention for the sake of compatibility with ultraproducts. It is still possible to represent unbounded structures (such as Banach spaces).
- Example: $(\mathbb{R}, d, +)$ where $d(x, y) = \frac{|x-y|}{1+|x-y|}$.

Definition

Two $\mathcal{L}$-structures M and N are *isomorphic* if there is **an isometry** $f : M \to N$ that respects all constant, relation, and function symbols of $\mathcal{L}$.

Ultraproducts and ultrapowers

- Fix a family $(M_i, d_i)_{i \in I}$ of $\mathcal{L}$-structures and an ultrafilter $\mathcal{F}$ on I.
- Consider the set $\prod_{i \in I} M_i$ and the extended pseudo-metric

$$d_{\mathcal{F}}(a, b) = \lim_{i \to \mathcal{F}} d_i(a(i), b(i)).$$

- **If we're lucky,** the equivalence relation $a \sim b \Leftrightarrow d_{\mathcal{F}}(a, b) = 0$ will respect relation and function symbols. If the metric and relations are also bounded, we can regard $M_{\mathcal{F}} := \prod_{i \in I} M_i / \sim$ as an $\mathcal{L}$-structure with metric $d = d_{\mathcal{F}}$. When this happens, relations agree with limit: $R^{M_{\mathcal{F}}}(a) = \lim_{i \to \mathcal{F}} R^{M_i}(a_i)$.
- $M_{\mathcal{F}}$ is the *ultraproduct of* $(M_i)_{i \in I}$ *(using* $\mathcal{F}$*)*.
- If $M_i = M$ for all $i \in I$, this is an *ultrapower*, written $M^{\mathcal{F}}$. There is a natural embedding $\iota : M \to M^{\mathcal{F}}$ taking a to the $\sim$-class of the constant function a.
- Example of finite approximation intuition: If $M = (\mathbb{R}, d = \frac{|x-y|}{1+|x-y|}, +)$, then $M^{\mathcal{F}}$ will contain an element a satisfying $d(a, 0) = 1$.

Doing continuous model theory backwards

Fix a class $\mathbb{K}$ of (metric) $\mathcal{L}$-structures that is closed under ultraproducts. In particular, **all ultraproducts must exist.**

Definition

Given $\mathcal{L}$-structures M, N and n-tuples $\bar{a} \in M$ and $\bar{b} \in N$, we say that $(M, \bar{a})$ and $(N, \bar{b})$ have the same *n-type*, and we write $(M, \bar{a}) \equiv (N, \bar{b})$, if there are ultrafilters $\mathcal{F}$ and $\mathcal{G}$ and an **(isometric)** isomorphism $f : M^{\mathcal{F}} \to N^{\mathcal{G}}$ such that $f(\iota(\bar{a})) = \iota(\bar{b})$.

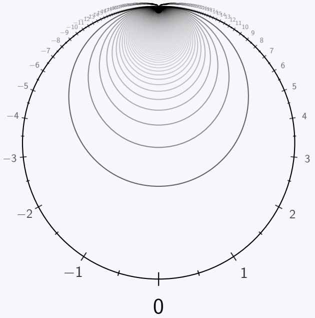
- For any $(M, \bar{a})$ and ultrafilter $\mathcal{F}$, we have that $(M, \bar{a}) \equiv (M^{\mathcal{F}}, \iota(\bar{a}))$.
- $\equiv$ is an equivalence relation. We write $S_n(\mathbb{K})$ for the $\equiv$-classes of n-tuples, which we call n-types. $\mathrm{tp}(M, \bar{a})$ is the $\equiv$-class of $(M, \bar{a})$.

Type spaces in continuous logic

- $S_n(\mathbb{K})$ is a set.
- $\equiv$ is compatible with ultraproducts. This induces a topology on $S_n(\mathbb{K})$ which is compact and Hausdorff (i.e., $S_n(\mathbb{K})$ is the Gelfand dual of a commutative C*-algebra). $S_n(\mathbb{K})$ **is not generally totally disconnected.**
- **An *n*-formula is a continuous function $\varphi : S_n(\mathbb{K}) \to \mathbb{R}$.**
- We write an *n*-formula as $\varphi(\bar{x})$ or φ. We write $M \models \varphi(\bar{a}) \leq r$ to mean that $\varphi(\mathrm{tp}(M, \bar{a})) \leq r$. Likewise for $<$, $=$, etc. We also write $\varphi^M(\bar{a})$ for $\varphi(\mathrm{tp}(M, \bar{a}))$.
- A *complete theory* T is the 0-type of a structure. We write $\mathrm{Th}(M)$ for $\mathrm{tp}(M, \varnothing)$, and $M \models T$ for $\mathrm{Th}(M) = T$. We say that M is a *model* of T.
- A class $\mathbb{K}$ is $\mathrm{Mod}(T) := \{M : M \models T\}$ if and only if $\mathbb{K}$ is closed under ultraproducts and ultraroots and $S_0(\mathbb{K})$ is a singleton. We write $S_n(T)$ for $S_n(\mathrm{Mod}(T))$.

An example of a type space

- Given an $\mathcal{L}$-structure M and a set $A \subseteq M$, we write M_A for the $\mathcal{L}_A$-structure M with constants added naming each element of A. We write T_A for $\mathrm{Th}(M_A)$.
- We write $S_n(A)$ for $S_n(T_A)$. The map that takes $a \in A$ to $\mathrm{tp}(a)$ is always a continuous injection.
- Let $M = (R, d, +)$ as before, and let $\mathcal{F}$ be a (typical) ultrafilter. The type space $S_1(M^{\mathcal{F}})$ looks (roughly) like this:



Operations on formulas I

- For any continuous function $F : \mathbb{R}^k \to \mathbb{R}$, we write $F(\varphi_1, \ldots, \varphi_k)$ for the obvious composition of functions (e.g., $\max(\varphi, \psi)$).
- If $(\varphi_i)_{i<\omega}$ is a **uniformly convergent sequence of n-formulas,** we write $\lim_{i\to\infty} \varphi_i$ for the limit.
- There is a natural map $\pi : S_{n+1}(\mathbb{K}) \to S_n(\mathbb{K})$ which takes $\mathrm{tp}(M, \bar{a}b)$ to $\mathrm{tp}(M, \bar{a})$.
- This map is continuous and open. This implies that for any $(n+1)$-formula φ, the function

$$\psi(p) = \inf\{\varphi(q) : \pi(q) = p\}$$

is continuous on $S_n(\mathbb{K})$. We write this formula as $\inf_y \varphi(\bar{x}, y)$. We also write $\sup_y \varphi$ for $-\inf_y -\varphi$.

Operations on formulas II

- The suggestion of the inf notation is correct: $M \models \inf_y \varphi(\bar{a}, y) \leq r$ if and only if $\inf\{\varphi^M(\bar{a}, b) : b \in M\} \leq r$.
- Basic expressions involving variables, symbols in the language, and the metric are equivalent to formulas (e.g., there is a formula $\varphi(x_1 x_2)$ such that for any M, $\varphi^M(ab) = d(a, b)$).
- The operations on this slide and the last generate all n-formulas for every n.

Some model theory

Elementary substructures

Basic model theory typically generalizes to continuous logic (until it doesn't).

Definition

A substructure N of a structure M is an *elementary substructure* if $N_N \equiv M_N$. We write $N \preceq M$ to signify this.

Prototypical example is M in $M^{\mathcal{F}}$.

Theorem (The Löwenheim-Skolem theorem)

For any structure M and $A \subseteq M$, there is $N \preceq M$ with $A \subseteq N$ such that the density character of N is no more than $|A| + |\mathcal{L}|$.

A general theme in continuous logic is that the correct way to count is density character (where compact is 'finite'). In model theory we often want to count types, so we need a metric on type space.

The metric on type space

For any type space $S_n(T)$, we get an (extended) metric

$$d(p,q) = \inf\{d^M(\bar{a},\bar{b}) : \bar{a},\bar{b} \in M \models T,\ \mathrm{tp}(\bar{a}) = p,\ \mathrm{tp}(\bar{b}) = q\}.$$

This metric has a lot of compatibility with the normal (compact) *logic topology*:

- The topology induced by d refines the logic topology.
- For any closed set F and any $\varepsilon > 0$, the set $F^{\leq \varepsilon} := \{p : d(p,F) \leq \varepsilon\}$ is closed.
- For any open set U and any $\varepsilon > 0$, the set $U^{<\varepsilon} := \{p : d(p,U) < \varepsilon\}$ is open.

The maps $\pi : S_{n+1}(T) \to S_n(T)$ and $\mathrm{tp} : M \to S_n(T)$ are 1-Lipschitz.

Definable sets

Definition

A closed set $D \subseteq S_n(T)$ is *definable* if the function $p \mapsto d(p, D)$ (on $S_n(T)$) is an n-formula (i.e., continuous).

For a closed set D, TFAE:

- D is definable.
- For each $\varepsilon > 0$, $D^{<\varepsilon}$ is open.
- For each $\varepsilon > 0$, $D \subseteq \operatorname{int} D^{<\varepsilon}$. (Interior is always in logic topology.)
- D is a Hausdorff limit of logically open sets.
- D admits relative quantification: For any $(m+n)$-formula $\varphi(\bar{x}, \bar{y})$, there is an m-formula $\psi(\bar{x})$ such that for any $M \models T$,

$$\psi^M(\bar{a}) = \inf\{\varphi^M(\bar{a}, \bar{b}) : \bar{b} \in M,\ \mathrm{tp}(\bar{b}) \in D\}.$$

Omitting types

Definition

A model $M \models T$ *realizes* $p \in S_n(T)$ if there is $\bar{a} \in M$ such that $\mathrm{tp}(\bar{a}) = p$.
A type $p \in S_n(T)$ is *atomic* if $\{p\}$ is definable.

Theorem (Omitting types)

($\mathcal{L}$ countable) For any $p \in S_n(T)$, TFAE:

- There is $M \models T$ not realizing p.
- There is an $\varepsilon > 0$ such that $B_{\leq\varepsilon}(p)$ has empty logical interior.
- p is not atomic.

Omitting partial types (i.e., closed subsets of $S_n(T)$) is generally very complicated (Farah and Magidor).

Prime and atomic models

Definition

$M \models T$ is a *prime model* if for every $N \models T$, there is $N_0 \preceq N$ isomorphic to M.
$M \models T$ is an *atomic model* if for every $\bar{a} \in M$, $\mathrm{tp}(\bar{a})$ is atomic.

Theorem

($\mathcal{L}$ countable) TFAE:

- T has a prime model.
- T has an atomic model.
- For every n, atomic types are (logically) dense in $S_n(T)$.

Categoricity I: The Ryll-Nardzewski theorem

Definition

For any cardinal κ, T is *κ-categorical* if T has a unique model of density character κ up to isomorphism.

Theorem

($\mathcal{L}$ countable) TFAE:

- T is ω-categorical.
- For each n, $S_n(T)$ is metrically compact (i.e., 'finite').
- For each n, every $p \in S_n(T)$ is atomic.
- Every $M \models T$ is 'approximately ω-saturated.'

In discrete logic, a countable theory T is ω-categorical if and only if $T_{\bar{a}}$ is ω-categorical for a finite tuple $\bar{a}$. This fails in continuous logic. There is an ω-categorical theory T such that T_a is not ω-categorical for some parameter a. Relatedly, Vaught's never-two theorem fails.

Categoricity II: Morley's theorem

A seminal result in discrete logic is Morley's theorem: If a countable complete theory T is κ-categorical for some uncountable κ, then it is λ-categorical for every uncountable λ.

Theorem (Ben Yaacov; Shelah and Usvyatsov)

($\mathcal{L}$ countable) Morley's theorem holds in continuous logic.

To generalize Morley's original proof, Ben Yaacov needed the correct generalization of ω-stability.

Definition

T is *κ-stable* if for any $M \models T$ and $A \subseteq M$ with $|A| \leq \kappa$, the metric density character of $S_1(A)$ is no more than κ.

- Ben Yaacov showed that uncountably categorical T are ω-stable.
- Much of discrete stability theory generalizes to continuous logic (until it doesn't).

Categoricity III: Baldwin and Lachlan

Baldwin and Lachlan sharpened the structural understanding of uncountably categorical theories. One important consequence of their work is that a countable complete theory T is uncountably categorical if and only it is ω-stable and has no Vaughtian pairs.

Definition

A *Vaughtian pair* is a pair M and N with $M \prec N$ such that for some definable set D, $D(M) = D(N)$ with $D(M)$ non-compact.

Theorem (Noquez)

If T is uncountably categorical, it has no Vaughtian pairs.

Counterexample (H.)

There is an ω-stable T with a strong form of no Vaughtian pairs that is not uncountably categorical.

Strongly minimal sets

A core aspect of Baldwin and Lachlan's structural understanding is that of a strongly minimal set.

Definition (Noquez)

A definable set $D \subseteq S_n(T)$ is *strongly minimal* if for any $M \models T$, $D(M)$ is non-compact and $S_n(D(M))$ contains a unique type not realized in $D(M)$. T is *strongly minimal* if $\{d(x,x) = 0\}$ is.

- Noquez asked whether there are any 'new' strongly minimal sets in continuous logic. (H.) $\mathrm{Th}(R, d, +)$ is strongly minimal and does not interpret any infinite discrete structures.
- Every discrete uncountably categorical theory has a strongly minimal set that 'controls' all models of the theory (via the no Vaughtian pairs condition).
- The theory of (the unit balls of) infinite-dimensional Hilbert spaces is uncountably categorical but has no strongly minimal sets.

Can we understand uncountably categorical theories?

- Strongly minimal sets are simple because they have a good notion of dimension, but so do Hilbert spaces.
- Hilbert spaces are essentially the only known example of this phenomenon.
- Shelah and Usvyatsov showed that any uncountably categorical expansion of (the unit ball of) a Banach space is somehow 'controlled' by a simple Hilbert space.

This leaves us with two questions:

- Can we find examples of uncountably categorical continuous theories that don't contain strongly minimal sets and are somehow fundamentally different from Hilbert spaces?
- Can we get a clear structural understanding of uncountably categorical continuous theories similar to that of Baldwin and Lachlan?

Thank you