How bad could it be?
The semilattice of definable sets in continuous logic

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April 21, 2023
CUNY Logic Workshop

Continuous logic

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- Elementary extensions: $\mathbb{R} \oplus \mathbb{Q}^{\oplus\kappa}$, where $\mathbb{Q}^{\oplus\kappa}$ has $\{0, 1\}$-metric.

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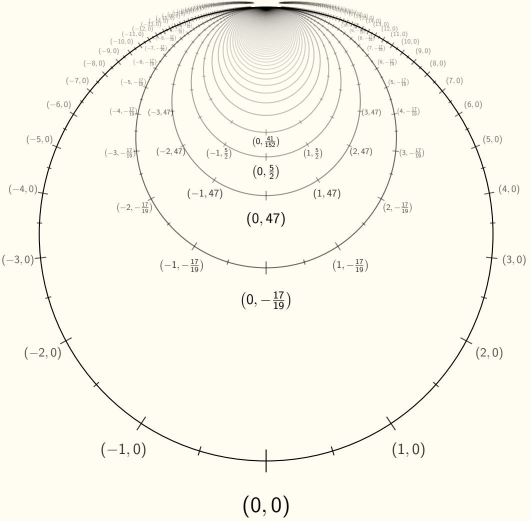
- (H.) Any compact topometric space (X, τ, ρ) with open metric ρ is isomorphic to $S_1(T)$ for some strictly stable T.

$S_1(\mathbb{R} \oplus \mathbb{Q})$

Unique non-algebraic type



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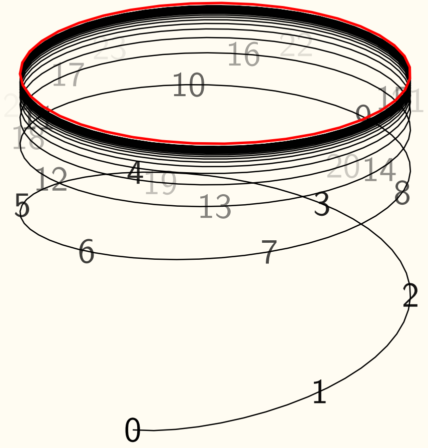
- If T is ω-stable, then $S_n(A)$ always has a basis of definable neighborhoods. (T is *dictionaric*.)

Many definable sets: $S_1(M)$, $M = (\mathbb{R}_{\geq 0}, \cos, \sin, d)$



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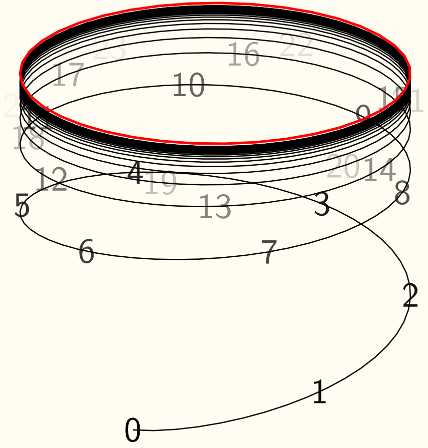

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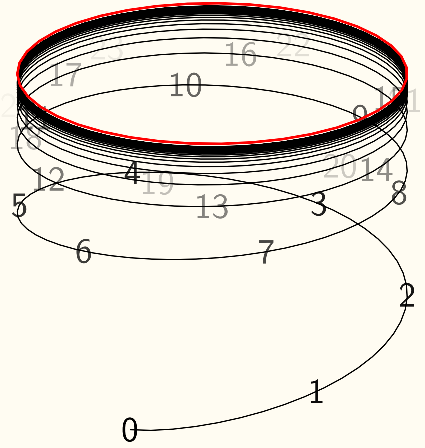
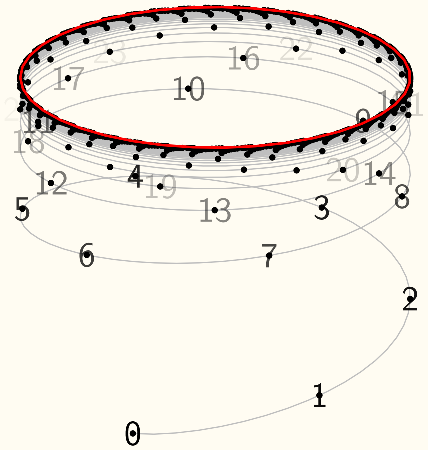


- $\mathrm{Th}(M)$ is ω-stable, so has many definable sets (e.g. $\{x : \cos(x) \in F\}$ for any closed F).
- Metric on non-algebraic types is (roughly) path metric.

Few definable sets: $S_1(N)$, $N = (\mathbb{N}, \mathrm{succ}, \cos, \sin, d)$



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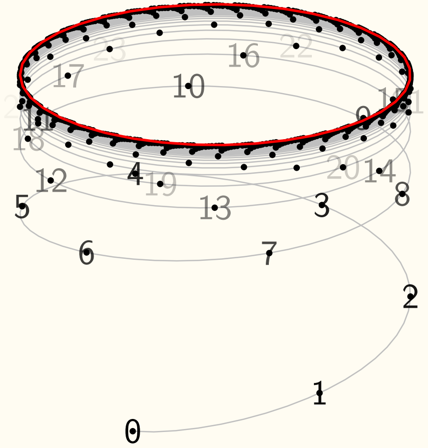
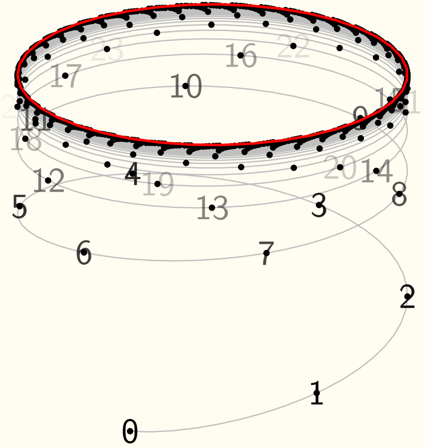


- $\mathrm{Th}(N)$ is superstable.



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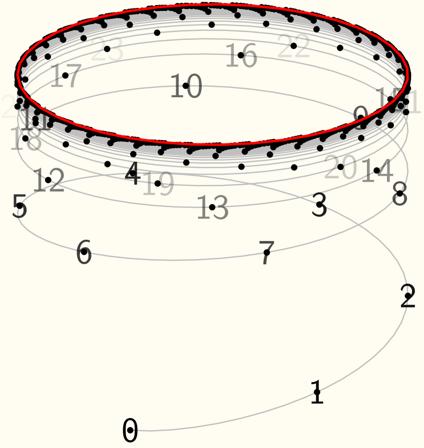
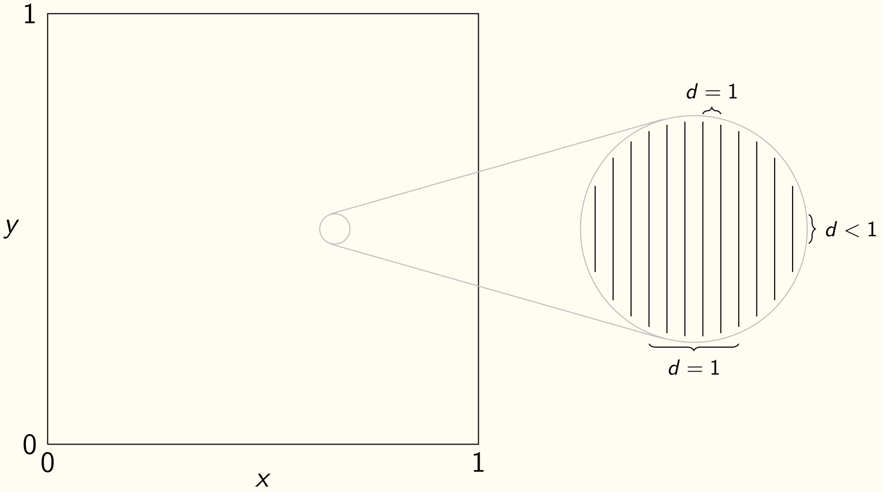


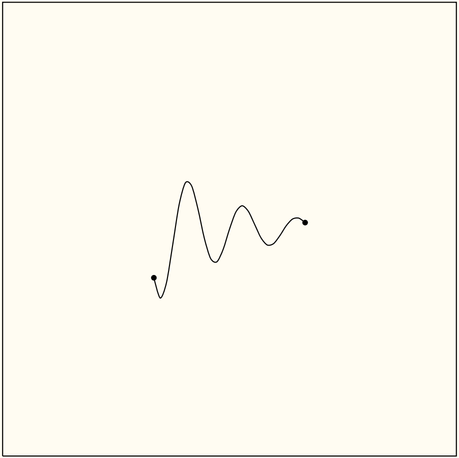
- $\mathrm{Th}(N)$ is superstable.
- Metric on non-algebraic types is discrete. Every definable set is either finite and algebraic or cofinite and co-algebraic.

Many but not enough I



$$d((x_0, y_0), (x_1, y_1)) = 1 \text{ if } y_0 \neq y_1.$$
$$d((x_0, y), (x_1, y)) = |x_0 - x_1|.$$

Many but not enough II



Closed F,

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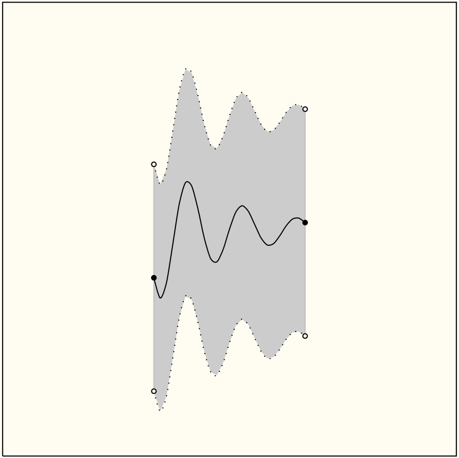

Closed F, with $F^{<\frac{1}{4}}$.

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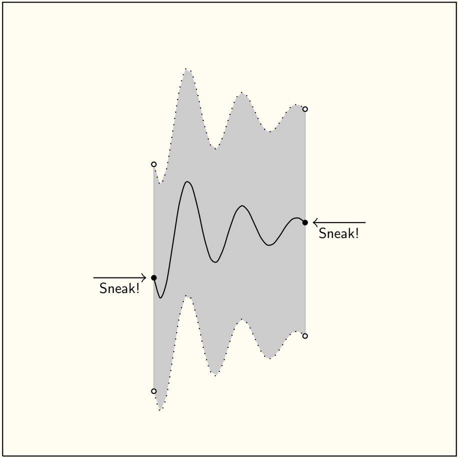
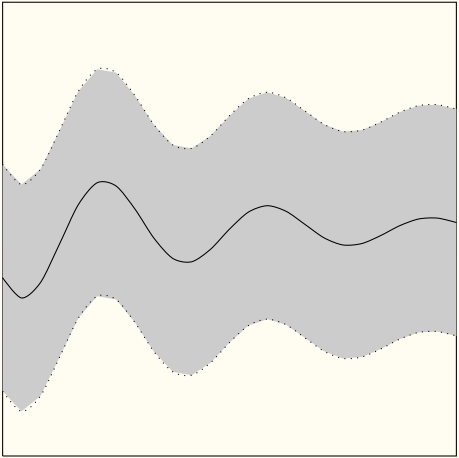


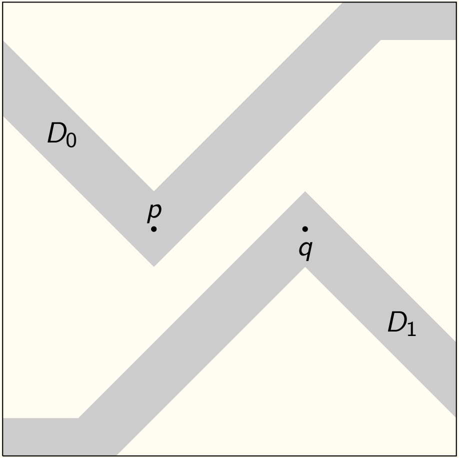
Closed F, with $F^{<\frac{1}{4}}$.
Not definable.

Many but not enough III



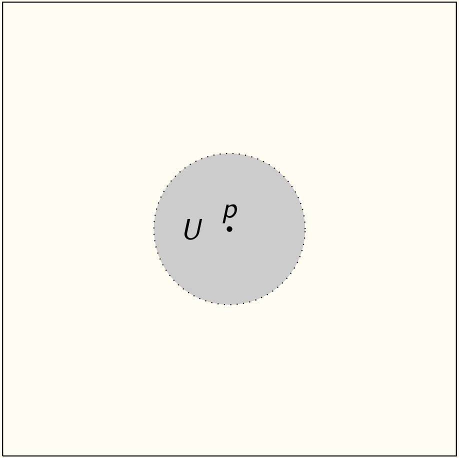
Definable set D, with $D^{<\frac{1}{4}}$.

Many but not enough IV



Almost any two points are separated by disjoint definable neighborhoods.

Many but not enough V



There is no non-empty definable D with $D \subseteq U$.

The semilattice of definable sets

Unions

Proposition

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Given a type space $S_n(T)$, the collection of definable subsets of it forms a bounded upper semilattice ($\varnothing$ and $S_n(T)$ are always definable) under unions.

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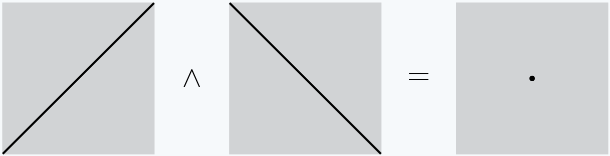
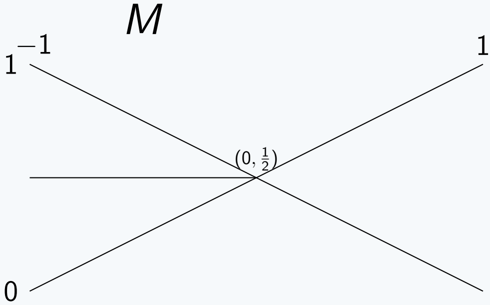
In square type space:



Prototypical example: Structure



- M is this subset of $[-1, 1] \times [0, 1]$.

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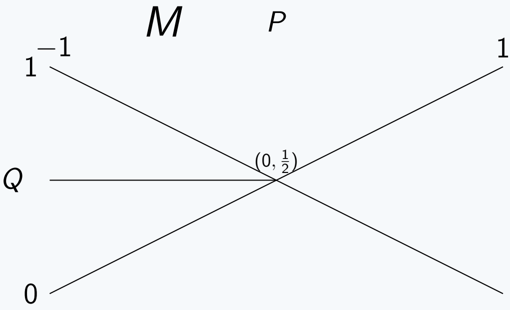

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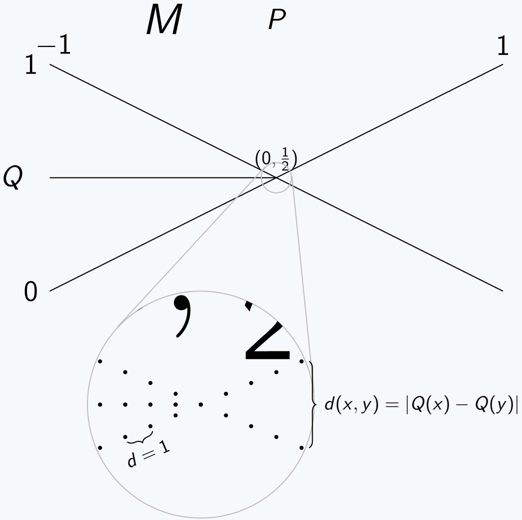

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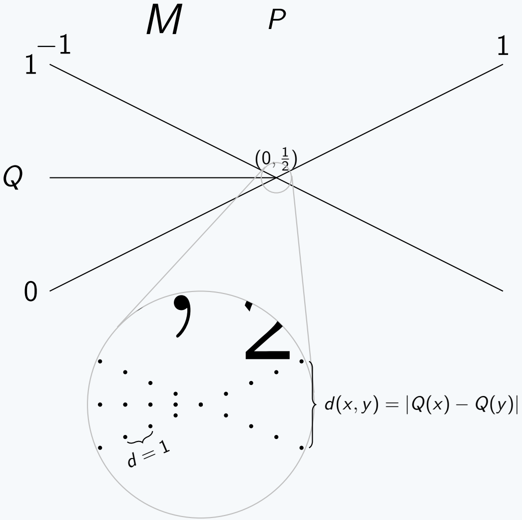

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- $d(x,y)=|Q(x)-Q(y)|$ if $P(x)=P(y)$ and is 1 otherwise.
- Let $T=\mathrm{Th}(M)$.

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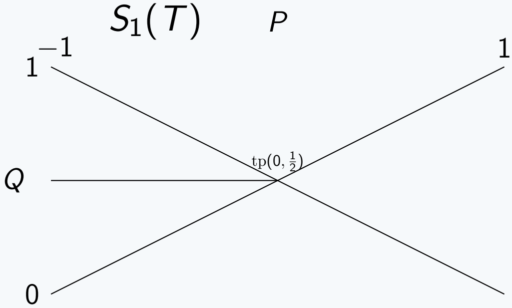

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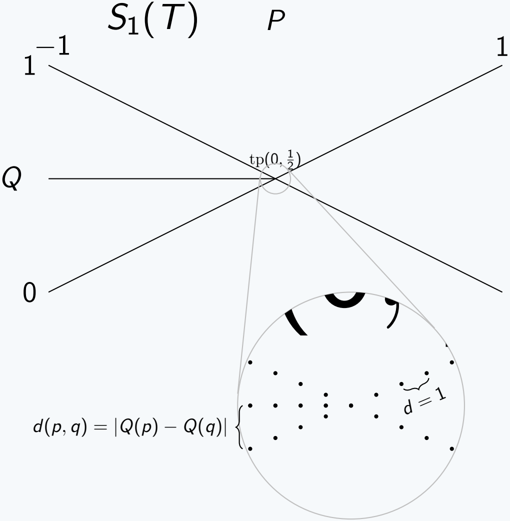

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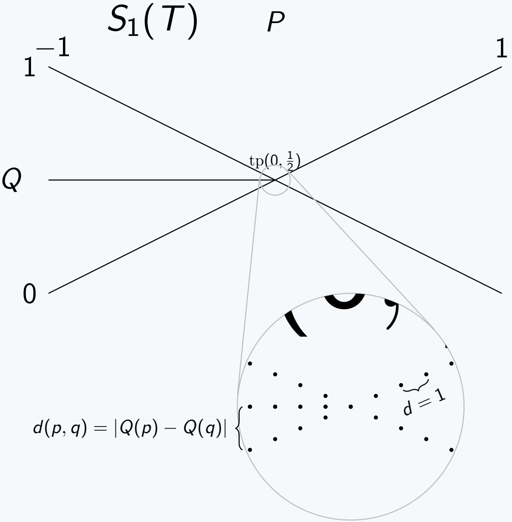
- Has precisely 22 definable sets.



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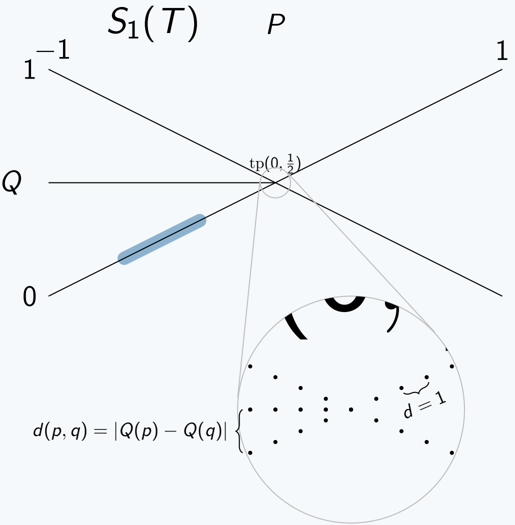

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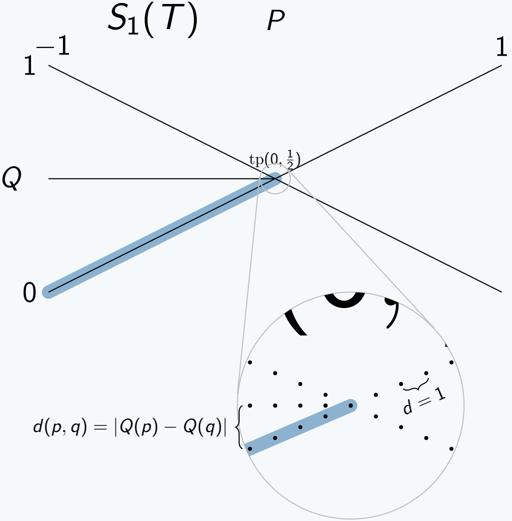

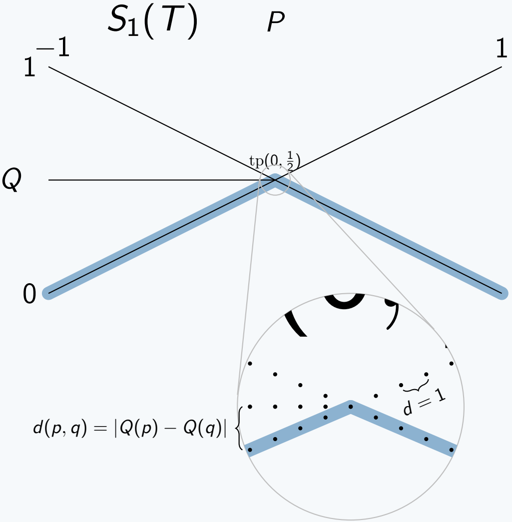

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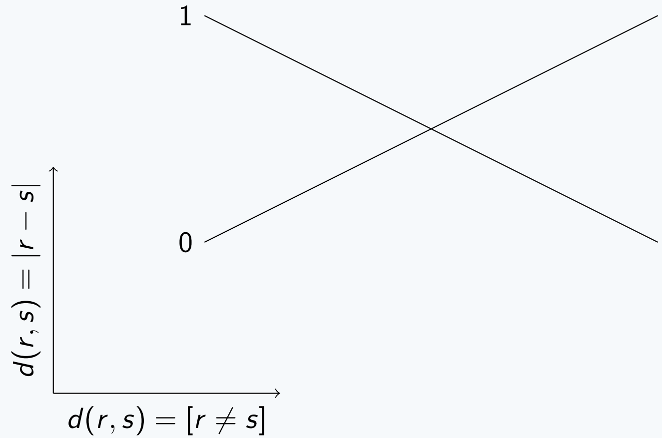
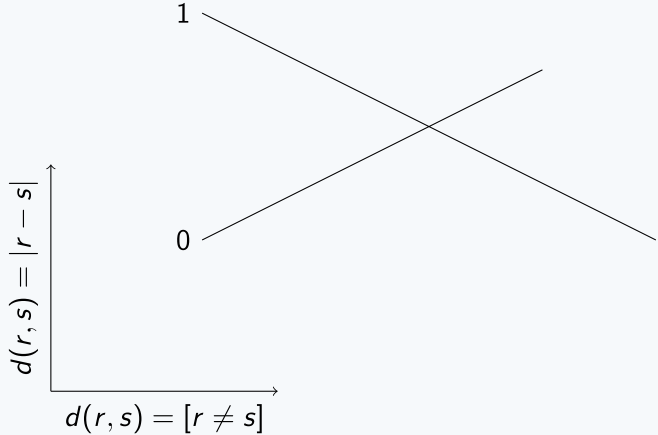
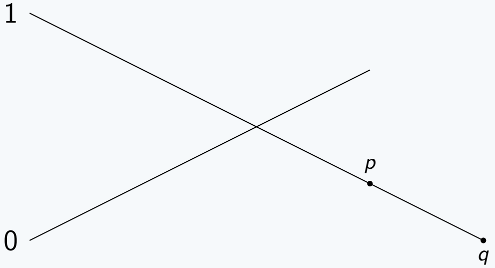
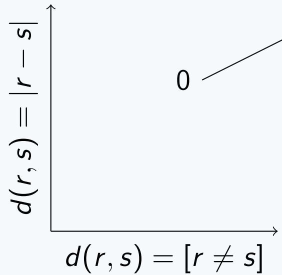


Wires



Wires





$$p(x) \models \inf_y \left|d(x,y) - \tfrac{1}{2}\right| = 0$$

$$q(x) \models \inf_y \left|d(x,y) - \tfrac{1}{2}\right| = \tfrac{1}{2}$$

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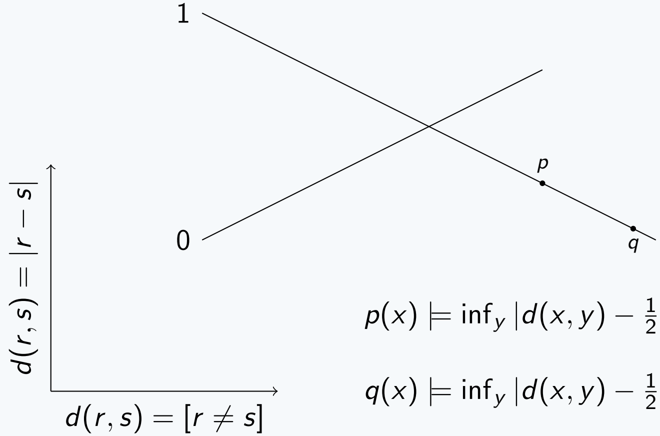

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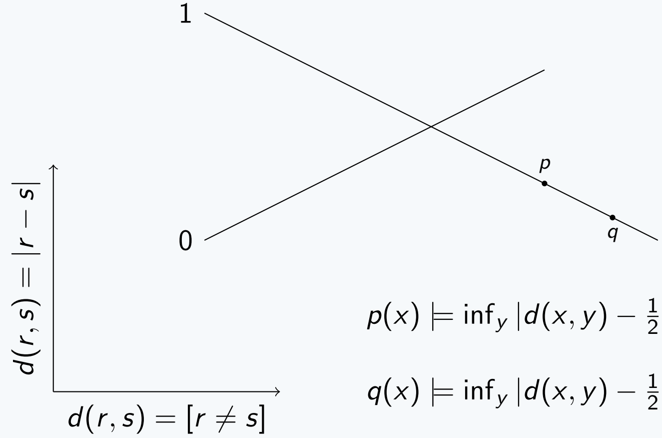

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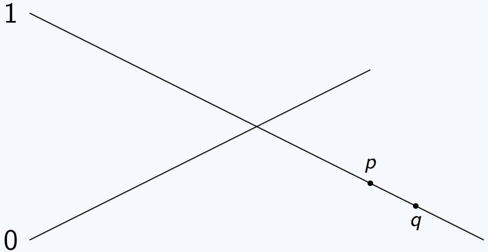

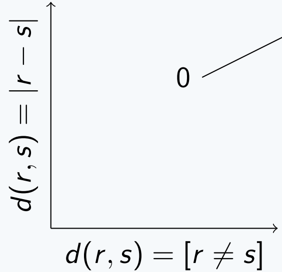

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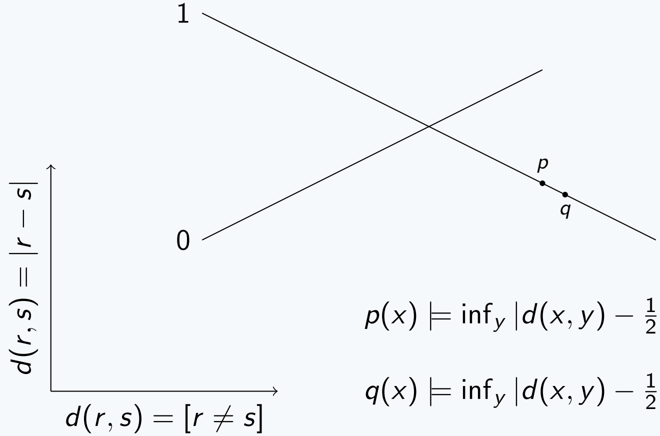

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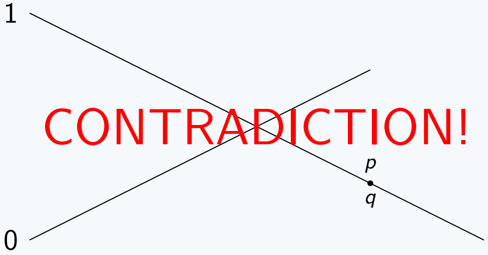

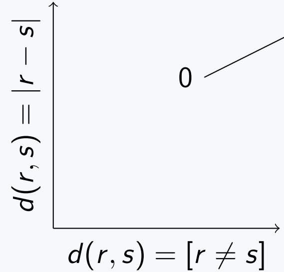

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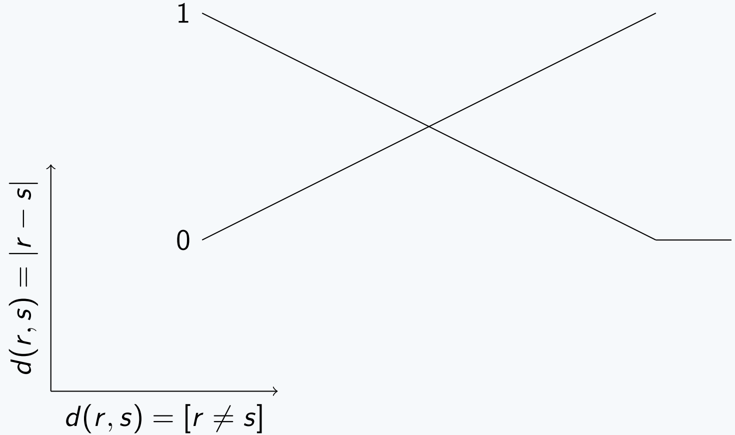

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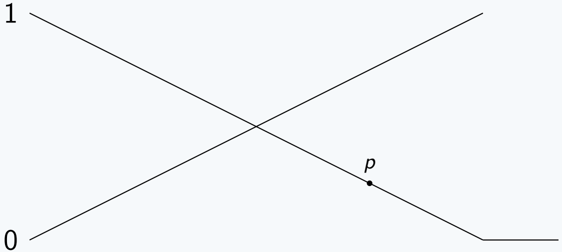

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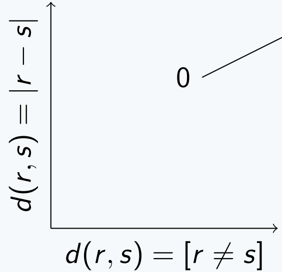
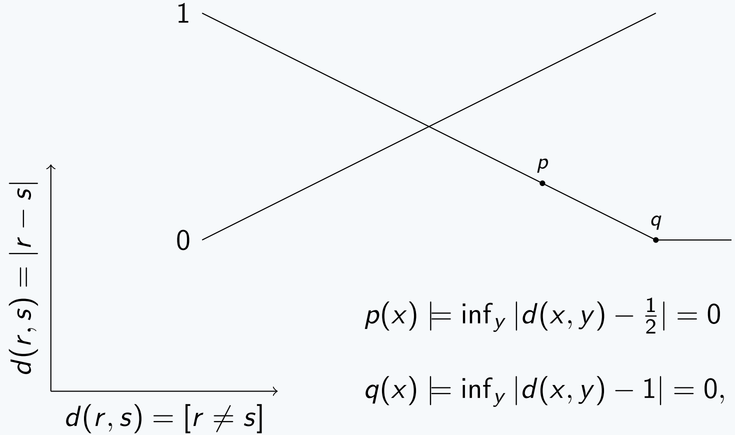


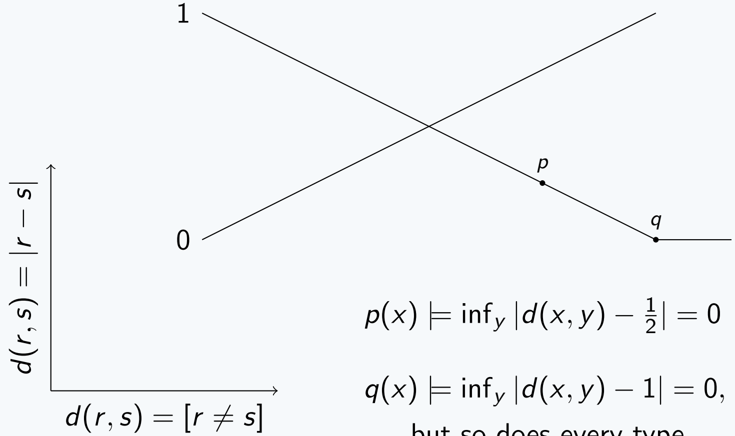
Wires



$$p(x) \models \inf_y |d(x,y) - \tfrac{1}{2}| = 0$$

$$q(x) \models \inf_y |d(x,y) - 1| = 0,$$

Wires



$p(x) \models \inf_y |d(x,y) - \frac{1}{2}| = 0$

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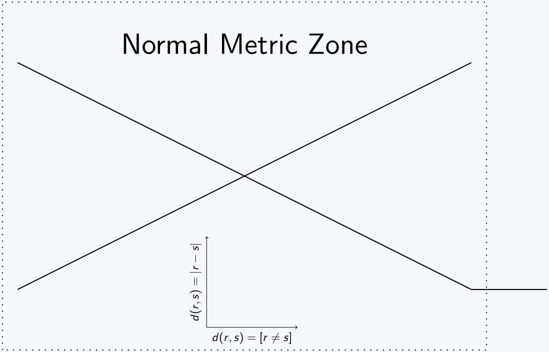
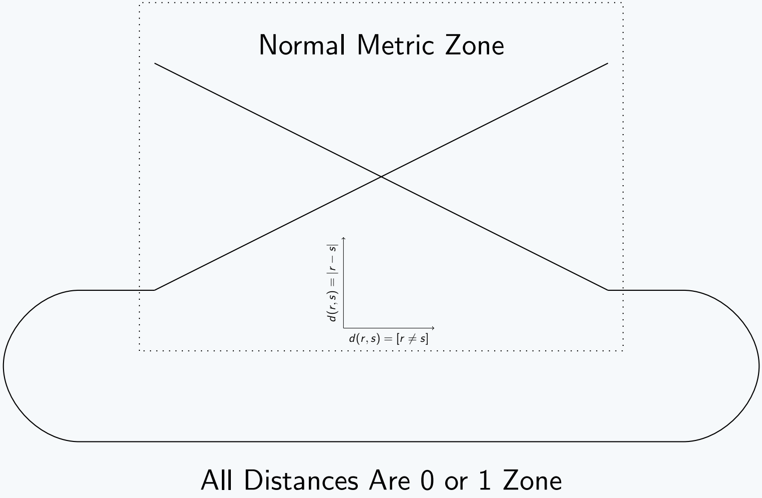
but so does every type.

Wires

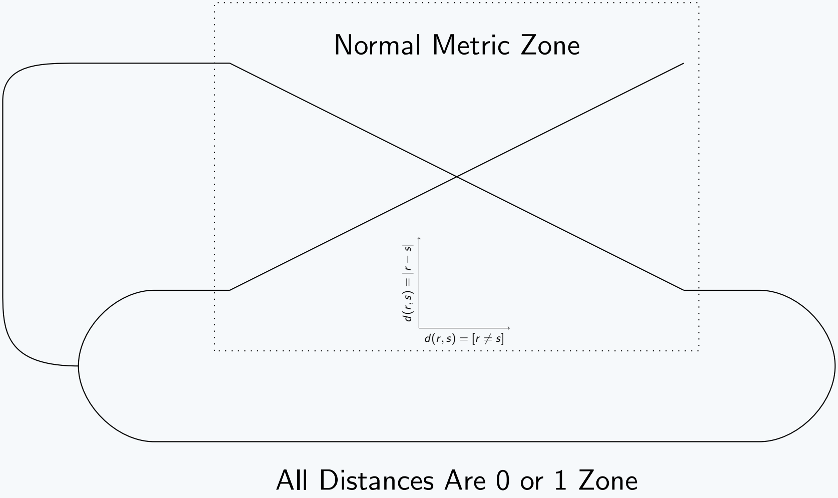


All Distances Are 0 or 1 Zone

Wires


Normal Metric Zone
$d(r,s) = |r - s|$
$d(r,s) = [r \neq s]$
All Distances Are 0 or 1 Zone

Wires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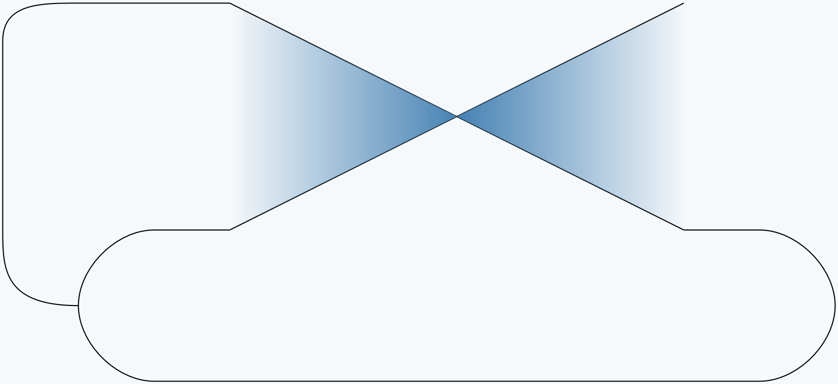
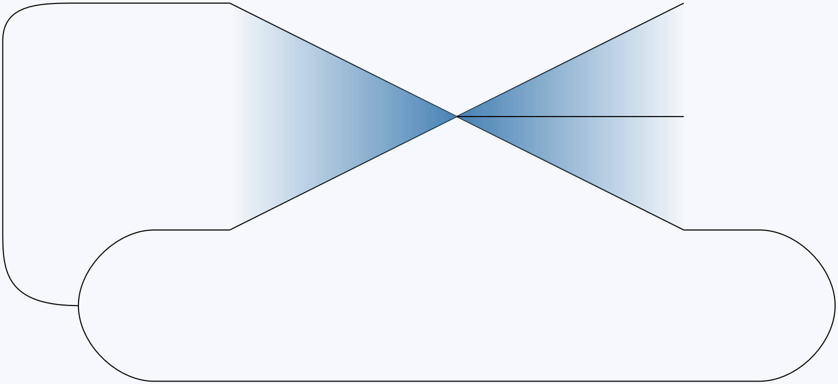
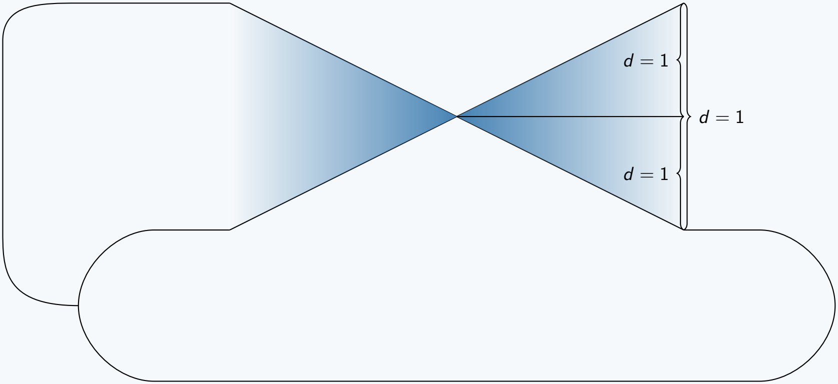


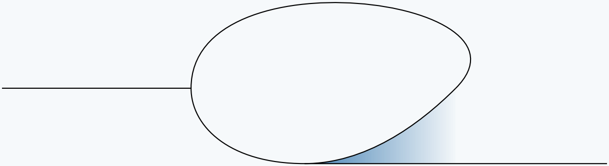
Wires



Wires

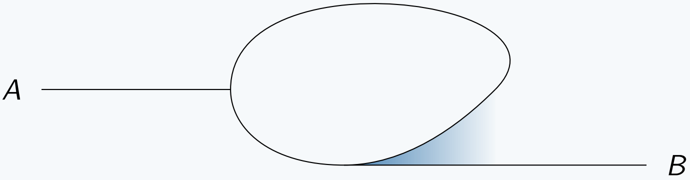


A diode



A diode

- Open Set U



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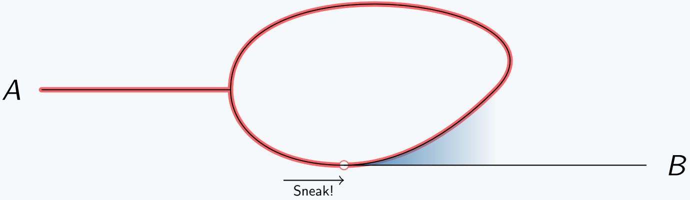
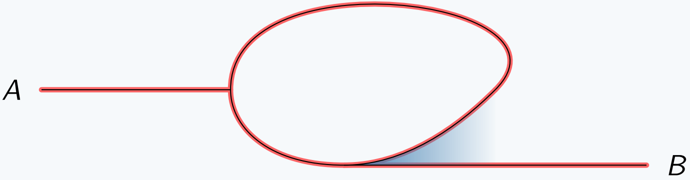


$S_1(T) \setminus U$ is not definable. X

A diode

- Open Set U



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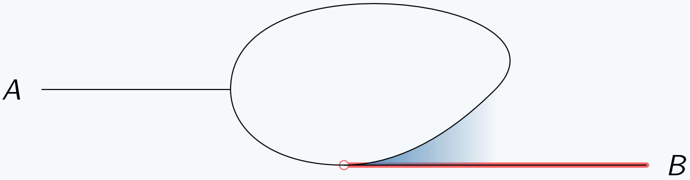

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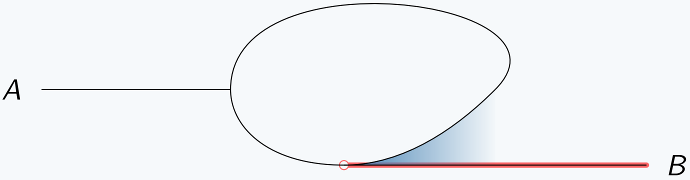
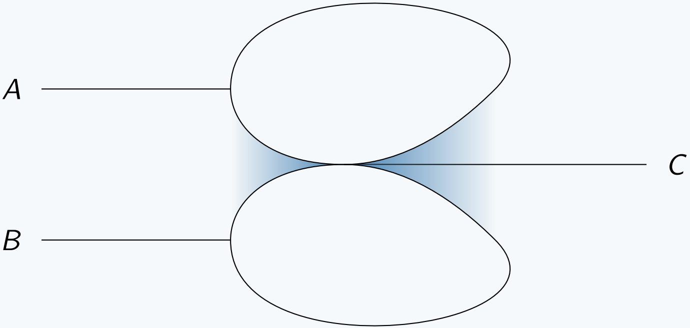


Roughly: $S_1(T) \setminus U$ is definable iff $A \in U \to B \in U$.

An AND gate

- Open Set U



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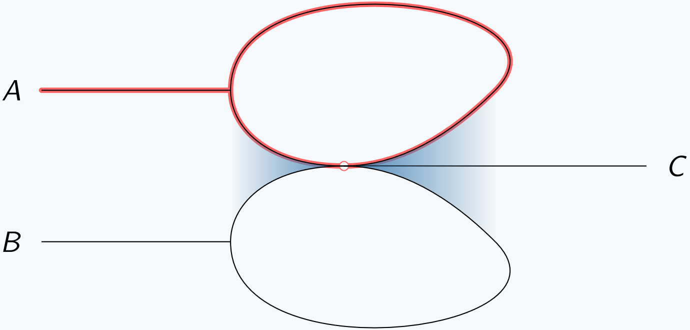

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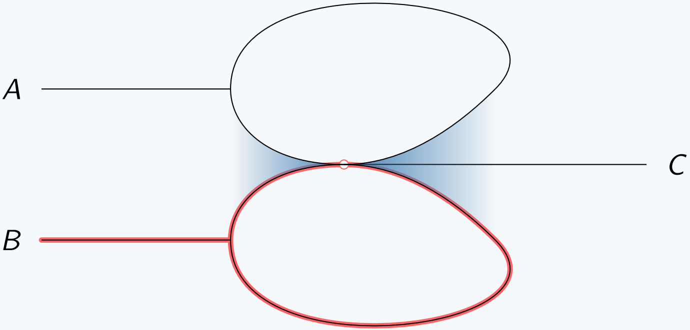

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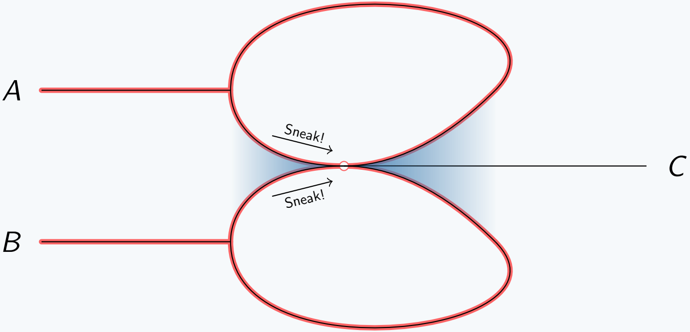

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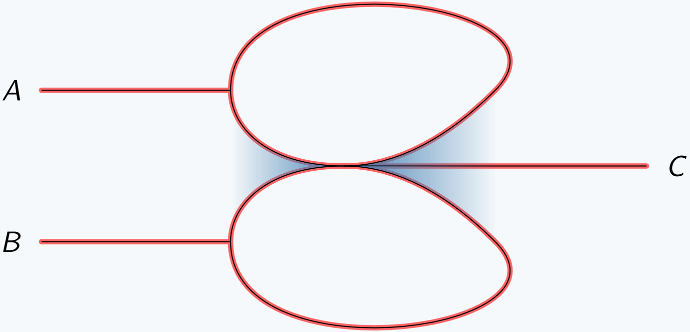

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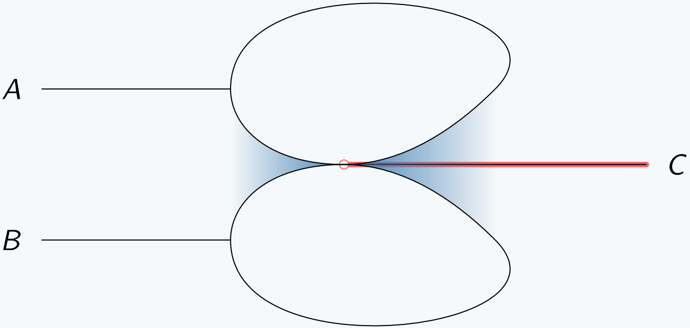

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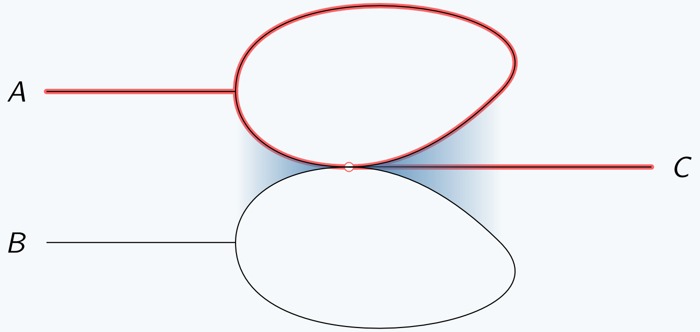

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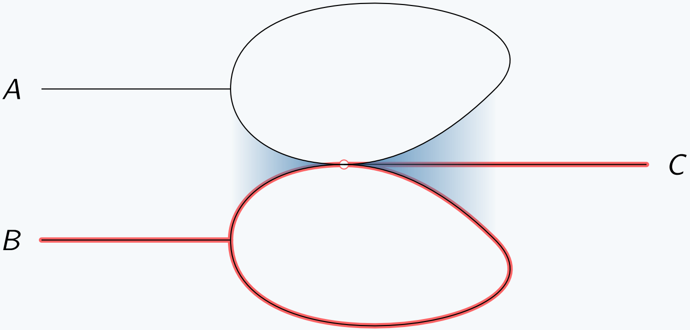
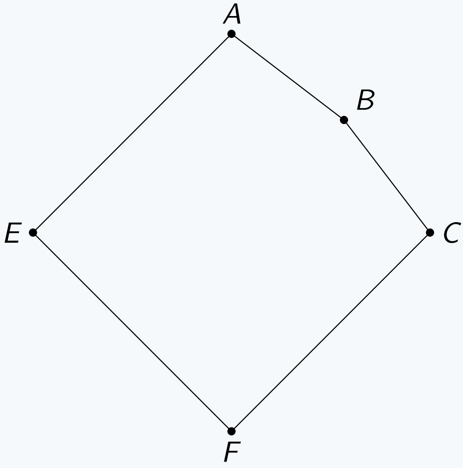


Roughly: $S_1(T) \setminus U$ is definable iff $A \in U \wedge B \in U \to C \in U$.

The construction

Take your favorite finite lattice with more than one element



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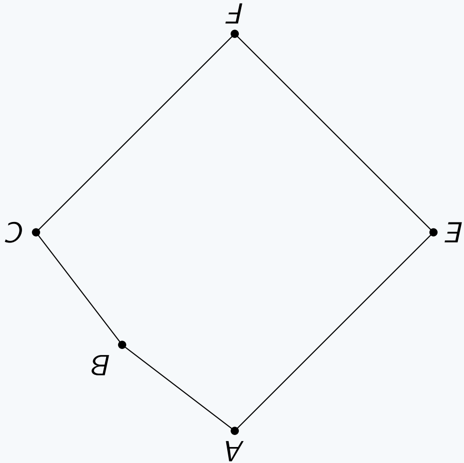
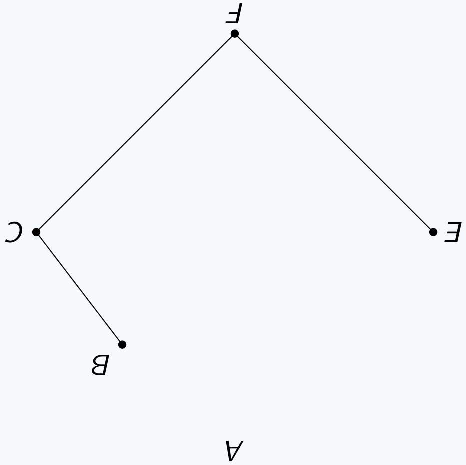
Take your favorite finite lattice with more than one element and flip it upside down.



Take your favorite finite lattice with more than one element and flip it upside down. Also cut the bottom off (point representing $\varnothing$ would be redundant).



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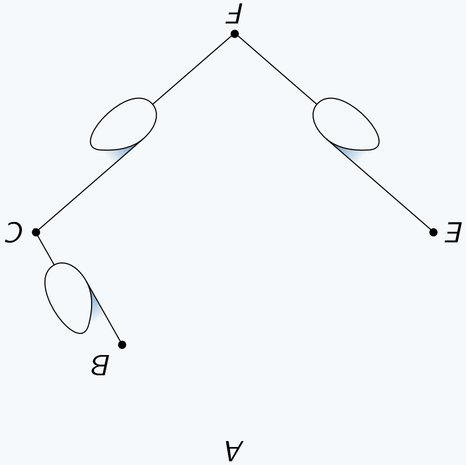
Diode for each pair with $x \leq y$



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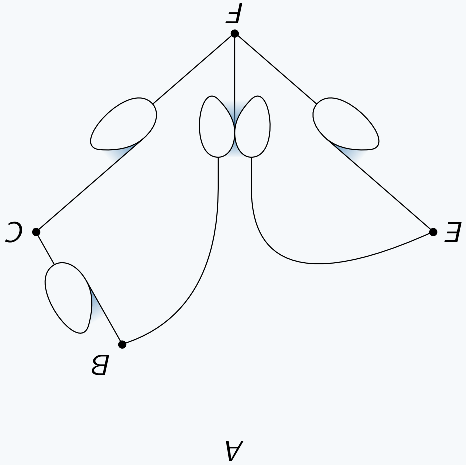
and AND gate for each triple with $x \leq y \vee z$



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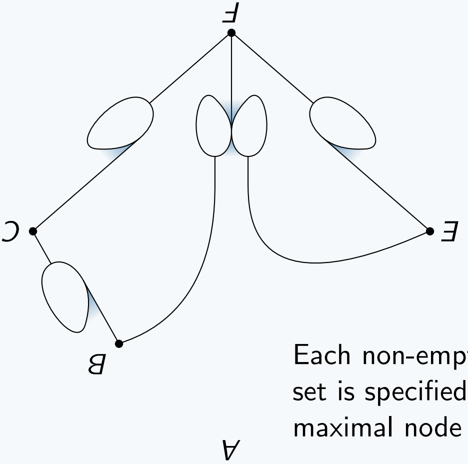

Each non-empty co-definable set is specified by the unique maximal node that it contains.

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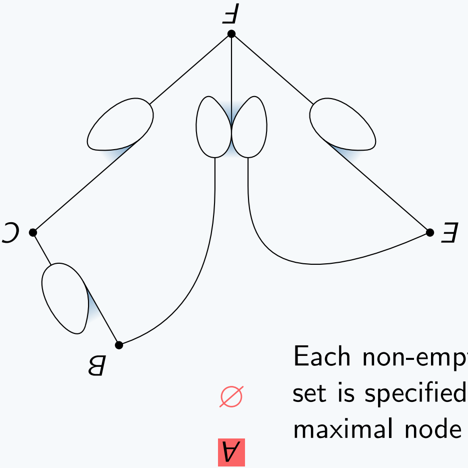

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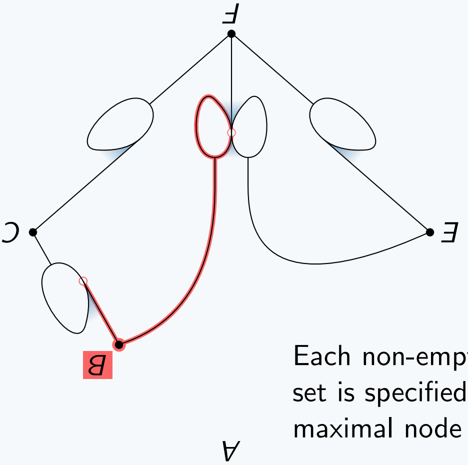

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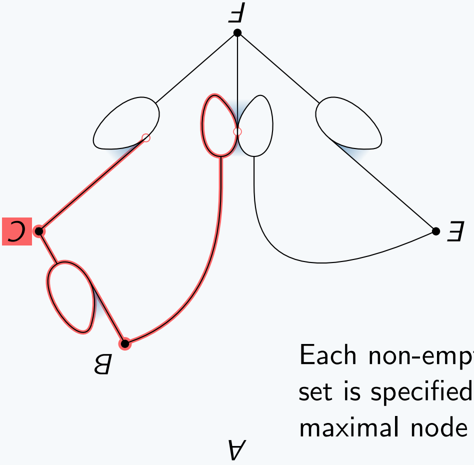

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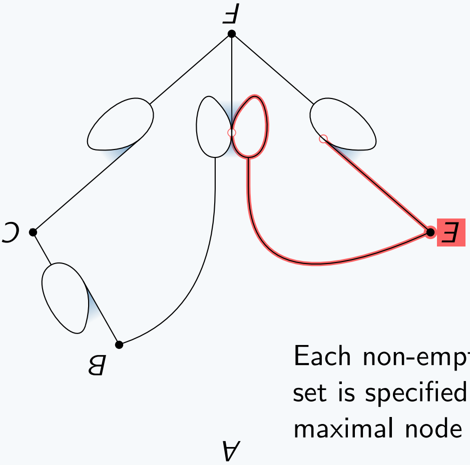

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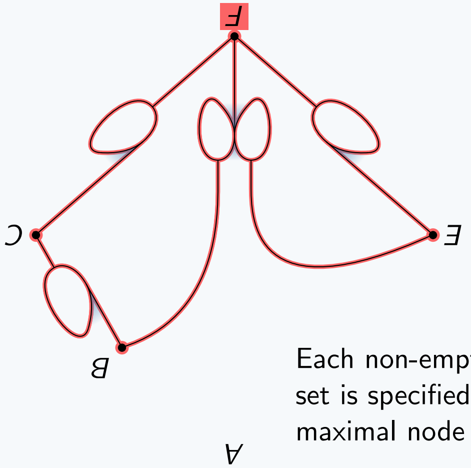

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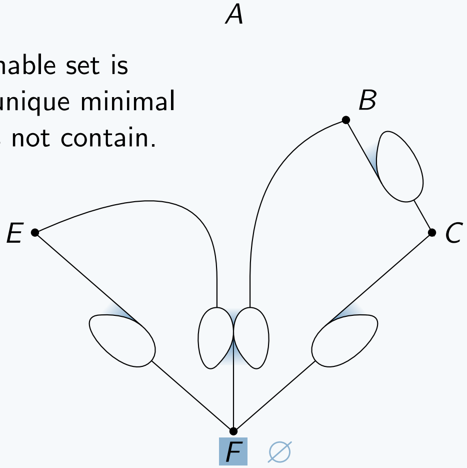
The construction

Take your favorite finite lattice with more than one element and flip it upside down. Also cut the bottom off (point representing ∅ would be redundant).

Dually, each definable set is specified by the unique minimal node that it does not contain.



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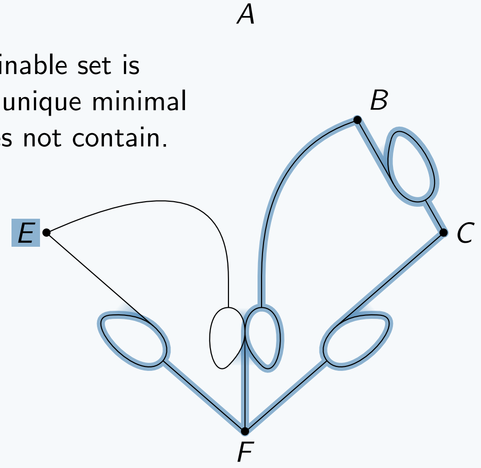

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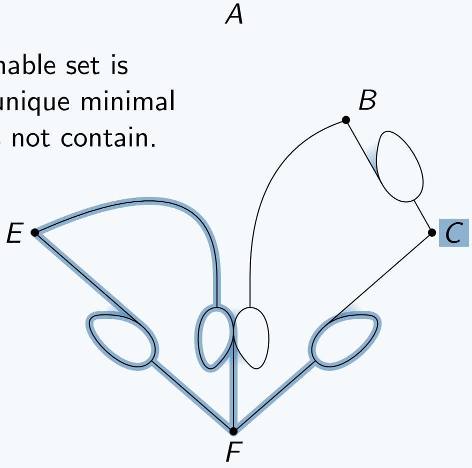

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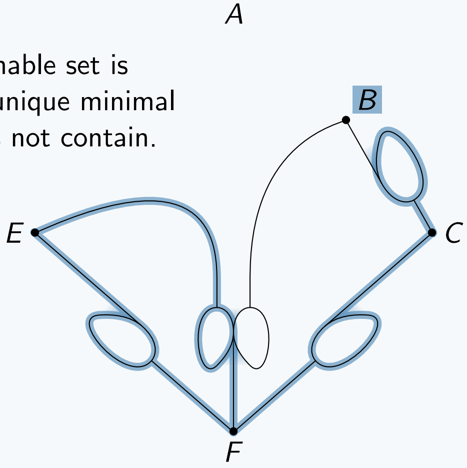

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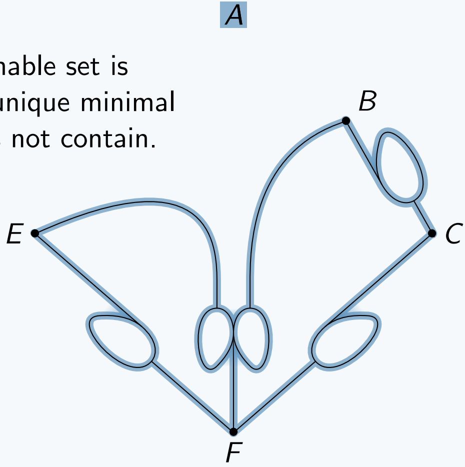

Topology

- The type spaces constructed here can always be embedded in $\mathbb{R}^3$ (topological graphs).

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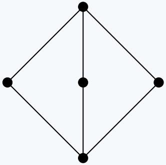

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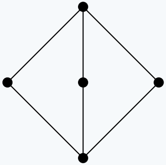

Question

What finite semilattices can be the partial order of definable sets in an $\mathbb{R}$-embeddable type space?

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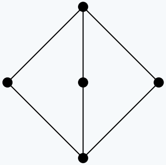

Question

What finite semilattices can be the partial order of definable sets in an $\mathbb{R}$-embeddable type space? An $\mathbb{R}^2$-embeddable type space?

Some infinite lattices

Which infinite semilattices can we have? I

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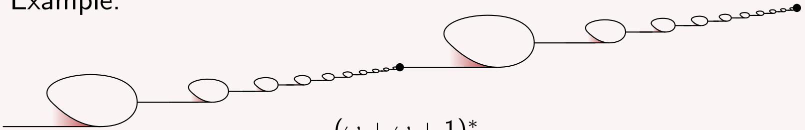
- Example:



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Thank you