Generic Stability and Randomizations

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Keisler Measures

- A *Keisler measure* $\mu(x)$ (over parameters A) is a function $\mu : \{\text{Boolean algebra of } A\text{-formulas in } x\} \to [0, 1]$ that obeys the standard laws of probability.
- Equivalently $\mu(x)$ can be thought of as a regular Borel probability measure on the relevant Stone space.
- 'A process to randomly choose a type.'
- Invented by Keisler to study forking in NIP theories.
- Measures are a generalization of types: Any type $p(x)$ induces a Dirac measure: $\delta_p(\varphi) = \begin{cases} 1 & \varphi \in p \\ 0 & \varphi \notin p \end{cases}$.

Invariant Types and Measures

Fix a first-order theory T with monster model $\mathfrak{C}$.

- A global type $p(x) \in S(\mathfrak{C})$ is *A-invariant* if each formula $\varphi(x, b)$'s membership in $p(x)$ only depends on $\varphi(x, y)$ and $\mathrm{tp}(b/A)$.
- For $q \in S_y(A)$, $F_p^{\varphi}(q) = \begin{cases} 1 & \varphi(x, b) \in p \text{ for some/any } b \models q \\ 0 & \varphi(x, b) \notin p \text{ for some/any } b \models q \end{cases}$.
- Invariant types play important roles in stable and NIP theories.
- A global measure $\mu(x)$ is *A-invariant* if $\mu(\varphi(x, b))$ only depends on $\varphi(x, y)$ and $\mathrm{tp}(b/A)$.
- For $q \in S_y(A)$, $F_\mu^{\varphi}(q) = \mu(\varphi(x, b))$ for some/any $b \models q$.

Morley Products

- Given A-invariant types $p(x)$ and $q(y)$, $p \otimes q(x, y)$ is the A-invariant type that corresponds to 'realizing q then realizing p over that.' We write $p^{\otimes 2} = p \otimes p$, $p^{\otimes 3} = p \otimes p \otimes p$, etc.
- $(a_i)_{i<\omega}$ is a *Morley sequence in p over B* if $a_{<\omega} \models p^{\otimes \omega} \restriction B$.
- Uniquely determined by the criterion $a_i \models p \restriction Ba_{<i}$ for all $i < \omega$.
- Given A-invariant measures $\mu(x)$ and $\nu(y)$, $\mu \otimes \nu(x, y)$ is the A-invariant measure that corresponds to 'realizing ν then realizing μ over that.'

Nice Invariant Types and Measures

The F_μ^φ specifying μ can be very nasty. Can also be nice:

- μ is *definable* if $F_\mu^\varphi : S_y(A) \to [0,1]$ is continuous for every $\varphi(x,y)$
- μ is *finitely satisfiable (in a small model)* if for any $\varphi(x,b)$ with $\mu(\varphi(x,b)) > 0$, there is $a \in A$ such that $\models \varphi(a,b)$
- *dfs* iff definable and finitely satisfiable.
- A type p is *generically stable* if whenever any Morley sequence in p behaves like a Morley sequence in a stable theory: If $a_{<\omega} \models p^{\otimes\omega}$, $\{i < \omega : \varphi(a_i, b)\}$ is always either finite or cofinite. Entails that p is the *average type* of any such sequence.

Some facts:

- T is stable $\Leftrightarrow$ every global type is generically stable.
- Generically stable $\Rightarrow$ dfs.
- (T NIP) dfs $\Rightarrow$ generically stable.

More Nice Invariant Measures

In NIP theories, dfs measures ought to be 'generically stable.'

Definition Idea (Sketch)

A measure μ is a *frequency interpretation measure* (or *fim* measure) if whenever '$a_i \models \mu{\restriction}Aa_{<i}$' for all $i < \omega$, the quantity $\frac{1}{n}|\{i < n : \varphi(a_i, b)\}|$ limits to $\mu(\varphi(x, b))$ with probability 1.

- A type is generically stable if and only if it is fim.
- fim $\Rightarrow$ dfs.
- (Hrushovski, Pillay, Simon) If T is NIP, dfs $\Rightarrow$ fim.

Randomizations

Given a theory T, there is a corresponding *randomization* T^R, a continuous theory formalizing the idea of 'random variables with values in a model of T.'

- Every (discrete or continuous) formula φ in T has a corresponding expected value formula $E[\varphi]$ in T^R.
- Types in $S(T^R)$ correspond precisely to measures in T (over $\varnothing$).

What about parameters? What about invariant types and measures?

- For any A-definable measure μ, there is a unique corresponding $\varnothing$-definable type r_μ in $(T_A)^R$ satisfying $F_\mu^\varphi(q) = F_{r_\mu}^{E[\varphi]}(\delta_q)$. (Just extend defining schema linearly.)
- (Conant, Gannon, H.) There is a dfs type p with the property that r_p is not finitely satisfiable in any small set.
- fim and generic stability?

fim and Generic Stability in the Randomization I

For any definable measure $\nu(x)$, let

$$\chi^{\varphi}_{\nu,n}(x_1 \dots x_n) = \sup_y \left| \frac{1}{n}(\underbrace{\varphi(x_1, y) + \cdots + \varphi(x_n, y)}_{\substack{\text{True is 1.} \\ \text{False is 0.}}}) - F^{\varphi}_{\nu}(\mathrm{tp}(y/A)) \right|.$$

- $\chi^{\varphi}_{\nu,n}(\bar{x})$ measures how well $\bar{x}$ approximates the behavior of $\nu(x)$ on average for the formula $\varphi(x, y)$.
- $\chi^{\varphi}_{\nu,n}(\bar{x})$ is a formula in the sense of continuous logic (because $\nu(x)$ is definable).

fim and Generic Stability in the Randomization II

Fix an A-definable measure $\mu(x)$ in T. Let $r_\mu(x)$ be the corresponding type in $(T_A)^R$.

- $\mu(x)$ is fim iff for every $\varphi(x,y)$, $\lim_{n\to\infty}\int \chi^{\varphi}_{\mu,n}(\bar{x})d\mu^{\otimes n}(\bar{x}) = 0$.
- $r_\mu(x)$ is generically stable iff for every $\varphi(x,y)$ (from T), $\lim_{n\to\infty} r_\mu^{\otimes n}\left(\chi^{E[\varphi]}_{r_\mu,n}(\bar{x})\right) = 0$. (Uses QE down to $E[\varphi]$.)

Some calculation gives:

$$r_\mu^{\otimes n}\left(\chi^{E[\varphi]}_{r_\mu,n}(\bar{x})\right) = r_\mu^{\otimes n}\left(\sup_y \left| E\left[\frac{\varphi(x_1y)+\cdots+\varphi(x_ny)}{n} - F^{\varphi}_{\nu}(y)\right]\right|\right)$$

$$\int \chi^{\varphi}_{\mu,n} d\mu^{\otimes n} = r_\mu^{\otimes n}\left(\sup_y E\left[\left|\frac{\varphi(x_1y)+\cdots+\varphi(x_ny)}{n} - F^{\varphi}_{\nu}(y)\right|\right]\right)$$

First $\leq$ second by Jensen's inequality, so if μ is fim, then r_μ is generically stable,but will it reverse?

Thank you